Can time-varying risk of rare disasters explain aggregate stock market volatility?

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01.07.11

Risk of Rare Disasters

- Disaster risk was originally conceived of as a means of explaining the equity premium (Reitz 1988).

- Barro (2006) and Barro & Ursua (2008): A distribution for disaster risk can be calibrated based on international data.

  The risk is sufficient to explain the magnitude of the equity premium.

- These models predict that the volatility of stock returns equals the volatility of dividends, and that excess stock returns are unpredictable.

  Empirical studies contradict these implications.

Facts to be explained

- Stock return volatility exceeds that of dividends.

- If returns are more volatile than dividends, it must be because of changes in the price-dividend ratio. By definition:

$$R_{t+1} = \frac{P_{t+1} + D_{t+1}}{P_t} = \frac{\frac{P_{t+1}}{D_{t+1}} + 1}{\frac{P_t}{D_t}} \frac{D_{t+1}}{D_t}$$

- Why does the price-dividend ratio move around? Economic models offer two possible reasons:

  ⋆ Changing future discount rates
  ⋆ Changing future expected cash flows.

- In the data, the price-dividend ratio forecasts discount rates, but not cash flows.

- Moreover, government bill rates are relatively stable.

This paper

- The probability of a rare disaster is time-varying.
- The representative agent has recursive preferences with EIS $= 1$ and RRA > 1.
- The model can be solved in closed form (up to an indefinite integral).
- Recursive utility has qualitatively different implications than time-additive (power) utility.

I show that time-varying disaster risk offers a simple and parsimonious explanation of aggregate stock market behavior.

A consumption model with rare events

Endowment process (aggregate consumption):

$$dC_t = \mu C_{t^-}\,dt + \sigma C_{t^-}\,dB_t + (e^{Z_t} - 1)C_{t^-}\,dN_t.$$

- Diffusion $\mu C_{t^-}\,dt + \sigma C_{t^-}\,dB_t$ represents behavior of consumption during normal times.

- Poisson process N_t represents disasters.

- When a jump occurs, $e^{Z_t} - 1$ gives the percent change in consumption. $Z_t < 0$ has time-invariant distribution ν. E_ν denotes expectations taken with respect to this distribution.

- Jumps occur with time-varying intensity λ_t.

  $\lambda_t \approx$ probability of a jump over the next year.

  Process for λ_t: $d\lambda_t = \kappa(\bar{\lambda} - \lambda_t)\,dt + \sigma_\lambda \sqrt{\lambda_t}\,dB_{\lambda,t}$.

- B_t, $B_{\lambda,t}$, N_t and Z_t are assumed to be independent of one another.

Utility function

- Let V_t denote continuation utility.

- Continuous-time formulation of recursive utility (Duffie & Epstein 1992):

$$V_t = E_t \int_t^\infty f(C_s, V_s)\, ds,$$

  where

$$f(C, V) = \beta(1-\gamma)V \left(\log C - \frac{1}{1-\gamma} \log((1-\gamma)V) \right).$$

- $\beta =$ discount rate

- $\gamma =$ Relative risk aversion (RRA)

- The EIS is equal to 1

Value function

- Let $J(W_t, \lambda_t)$ denote the value function, where W_t is aggregate wealth. In equilibrium $J(W_t, \lambda_t) = V_t$.

- The solution for the value function:

$$J(W_t, \lambda_t) = \frac{W_t^{1-\gamma}}{1-\gamma} I(\lambda_t) = \frac{W_t^{1-\gamma}}{1-\gamma} e^{a+b\lambda_t},$$

  where

$$b = \frac{\kappa+\beta}{\sigma_\lambda^2} - \sqrt{\left(\frac{\kappa+\beta}{\sigma_\lambda^2}\right)^2 - 2\frac{E_\nu\left[e^{(1-\gamma)Z_t} - 1\right]}{\sigma_\lambda^2}}.$$

- $\gamma > 1 \Rightarrow e^{(1-\gamma)Z_t} > 1 \Rightarrow b > 0.$

  Therefore an increase in disaster risk reduces utility.

- In equilibrium, assets that fall in price when disasters become more likely must carry an extra risk premium.

Riskfree rate

The riskfree rate equals:

$$r_t = \beta + \mu - \gamma\sigma^2 + \lambda_t E_\nu \left[e^{-\gamma Z} \left(e^Z - 1 \right) \right].$$

- Standard diffusion term: $\beta + \mu - \gamma\sigma^2$
- Jump risk term: $\lambda_t E_\nu \left[e^{-\gamma Z} \left(e^Z - 1 \right) \right]$.
- $e^Z < 1$, so the jump risk term is negative.
- An increase in the probability of a disaster increases the agent's desire to save, thus lowering the riskfree rate.

Aggregate dividends

- Dividend process: $\log D_t = \phi \log C_t$ (leverage model of Abel, 1999).

- Ito's Lemma implies:

$$\frac{dD_t}{D_{t^-}} = \mu_D \, dt + \phi\sigma \, dB_t + (e^{\phi Z_t} - 1) \, dN_t,$$

where $\mu_D = \phi\mu + \frac{1}{2}\phi(\phi - 1)\sigma^2$.

- If $\phi > 1$, dividends fall by more than consumption in the event of a disaster.

The aggregate stock market

- State-price density

$$\pi_t = e^{\int_0^t f_V(C_s, V_s)\, ds} \beta^\gamma C_t^{-\gamma} I(\lambda_t)$$

- Let $F_t = F(D_t, \lambda_t)$ denote the price of the claim to dividends D_t. Then in equilibrium,

$$\begin{aligned} F(D_t, \lambda_t) &= E_t\left[\int_t^\infty \frac{\pi_s}{\pi_t} D_s\, ds\right] \\ &= D_t G(\lambda_t) \\ &= D_t \int_0^\infty \exp\left\{a_\phi(\tau) + b_\phi(\tau)\lambda\right\}\, d\tau, \end{aligned}$$

  where $a_\phi(\tau)$ and $b_\phi(\tau)$ have analytical expressions such that $b_\phi(\tau) < 0$ for all τ.

- The price-dividend ratio $G(\lambda_t)$ is decreasing in λ_t.

The equity premium

- Define the continuous-time analogue of the expected return:

$$r_t^e = \mu_{F,t} + \frac{D_t}{F_t} - \lambda_t E_\nu \left[e^{\phi Z} - 1 \right]$$

In equilibrium:

$$r_t^e - r_t = -\sigma_{\pi,t}\sigma_{F,t}^\top + \lambda_t E_\nu \left[\left(e^{-\gamma Z} - 1 \right) \left(1 - e^{\phi Z} \right) \right]$$

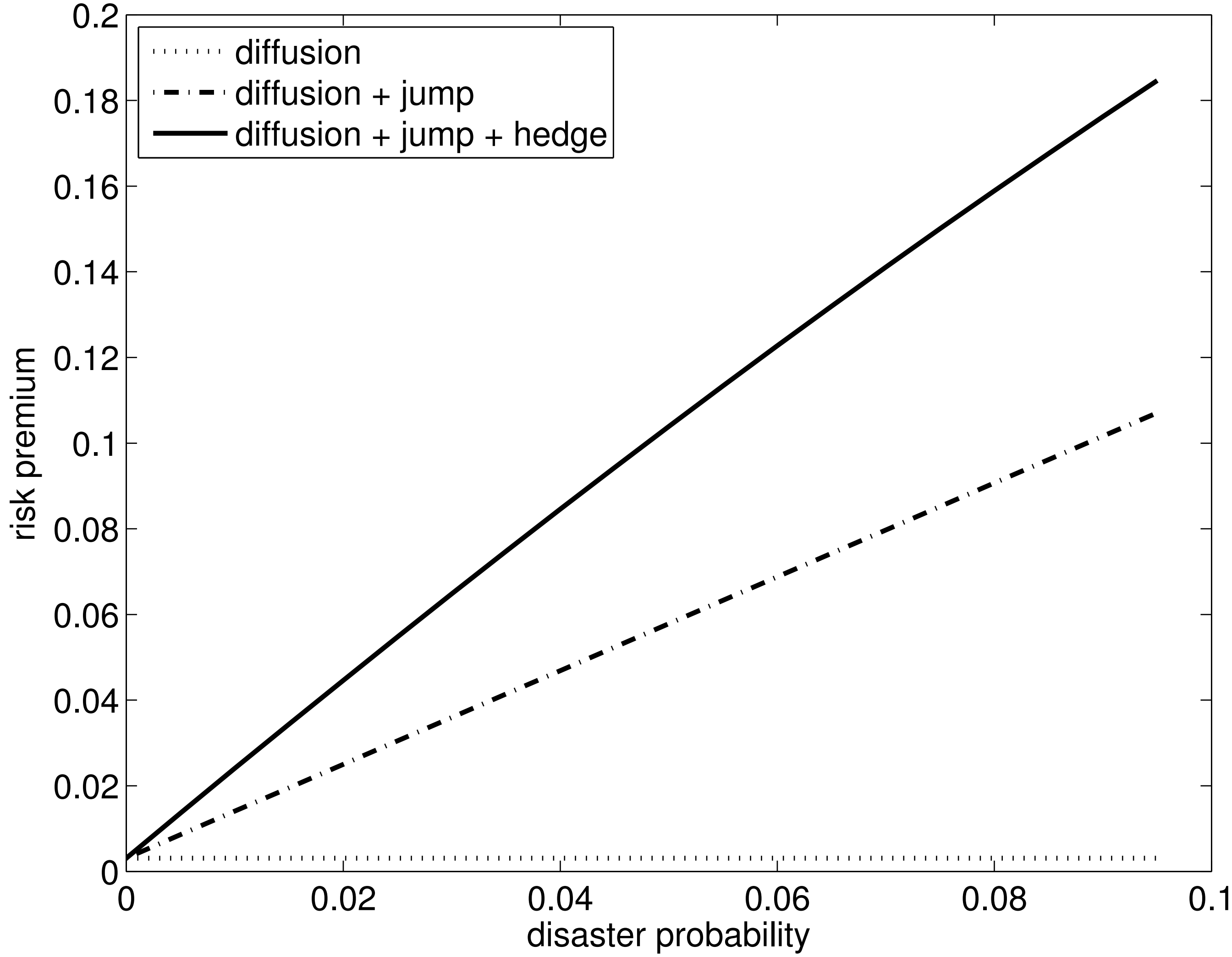
  - ⋆ The first term is the compensation for diffusion risk (which includes time-varying λ_t).
  - ⋆ The second term is the compensation for jump risk.
  - ⋆ Diffusion term for the state-price density:

$$\sigma_{\pi,t} \quad = \quad \left[-\gamma\sigma \quad b\sigma_\lambda \sqrt{\lambda_t} \right]$$

    $\Rightarrow$ variation in disaster risk is priced.
  - ⋆ Ito's Lemma implies that the equity premium equals

$$r_t^e - r_t = \phi\gamma\sigma^2 - \lambda_t b \sigma_\lambda^2 \frac{G'(\lambda_t)}{G(\lambda_t)} + \lambda_t E_\nu \left[\left(e^{-\gamma Z} - 1 \right) \left(1 - e^{\phi Z} \right) \right]$$

Decomposition of the equity premium



Power utility

- Utility: $V_t = E_t \int_t^\infty e^{-\beta s} \frac{C_s^{1-\gamma}}{1-\gamma}\, ds$

- Riskfree rate: $r_t = \beta + \gamma\mu - \frac{1}{2}\gamma(\gamma+1)\sigma^2 - \lambda_t E_\nu \left[e^{-\gamma Z} - 1\right]$

  Compare with the riskfree rate for recursive utility:

$$r_t = \beta + \mu - \gamma\sigma^2 - \lambda_t E_\nu \left[e^{-\gamma Z} - e^{(1-\gamma)Z}\right].$$

- Equity premium:

$$r_t^e - r_t = \phi\gamma\sigma^2 + \lambda_t E_\nu \left[(e^{-\gamma Z} - 1)(1 - e^{\phi Z})\right]$$

  Compare with the equity premium for recursive utility:

$$r_t^e - r_t = \phi\gamma\sigma^2 - \lambda_t b \sigma_\lambda^2 \frac{G'(\lambda_t)}{G(\lambda_t)} + \lambda_t E_\nu \left[(e^{-\gamma Z} - 1)\left(e^{\phi Z} - 1\right)\right]$$

- The price-dividend ratio is increasing in λ_t for $\gamma > \phi$.

Risk of default

- Disasters often coincide with partial default on government securities.

- Assume that, in the event of a disaster, there will be a partial default on government securities with probability q.

- For simplicity, assume the percent loss is equal to that of the consumption claim.

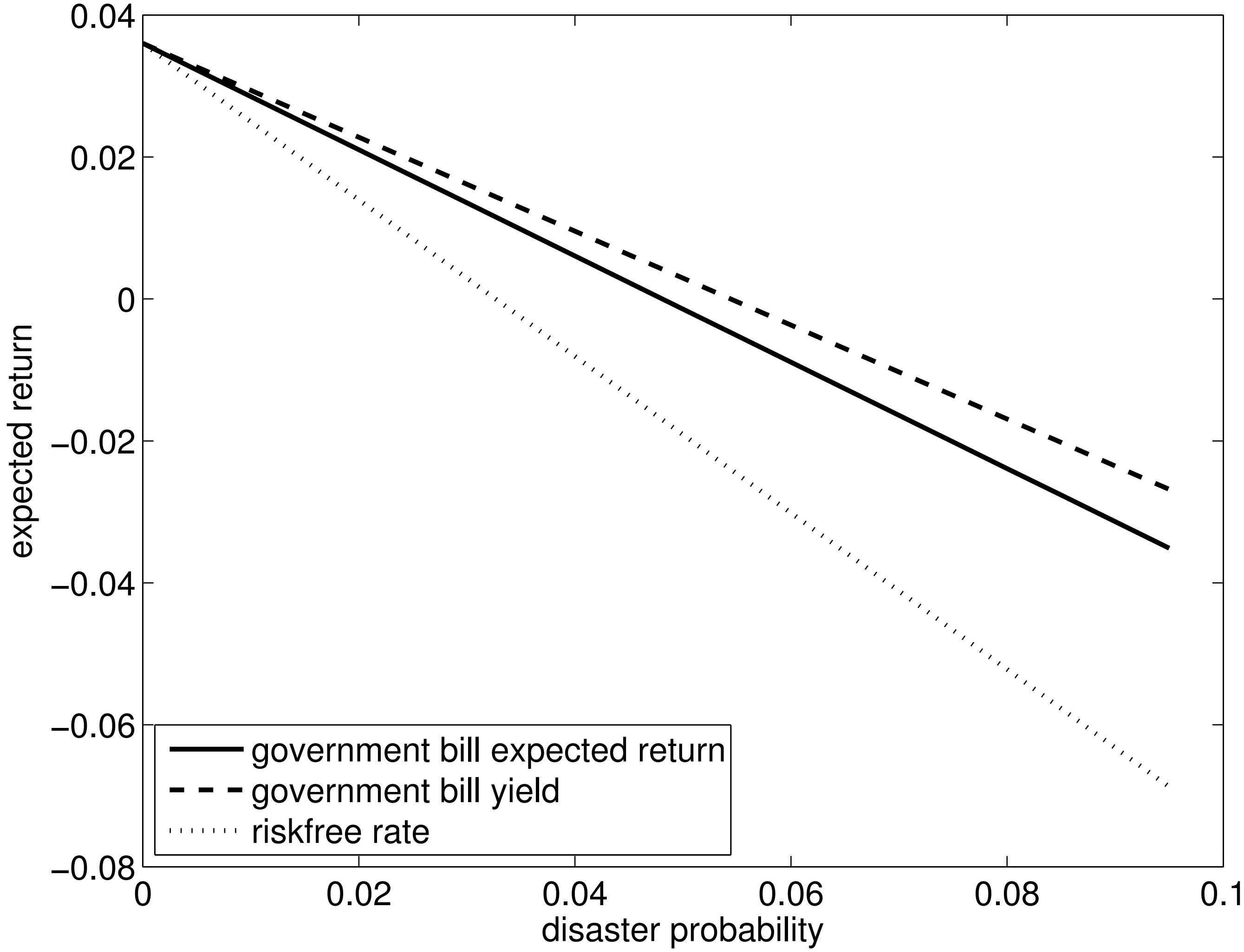
- Let r_t^L denote the return in the absence of default. In equilibrium,

$$r_t^L = r_t + \lambda_t E_\nu \left[e^{-\gamma Z} - 1 \right] - \lambda_t \left((1-q) E_\nu \left[e^{-\gamma Z} - 1 \right] + q E_\nu \left[e^{(1-\gamma) Z} - 1 \right] \right).$$

- The instantaneous expected return on government debt equals

$$r_t^b = r_t^L + \lambda_t q E_\nu \left[e^Z - 1 \right].$$

Government bill return

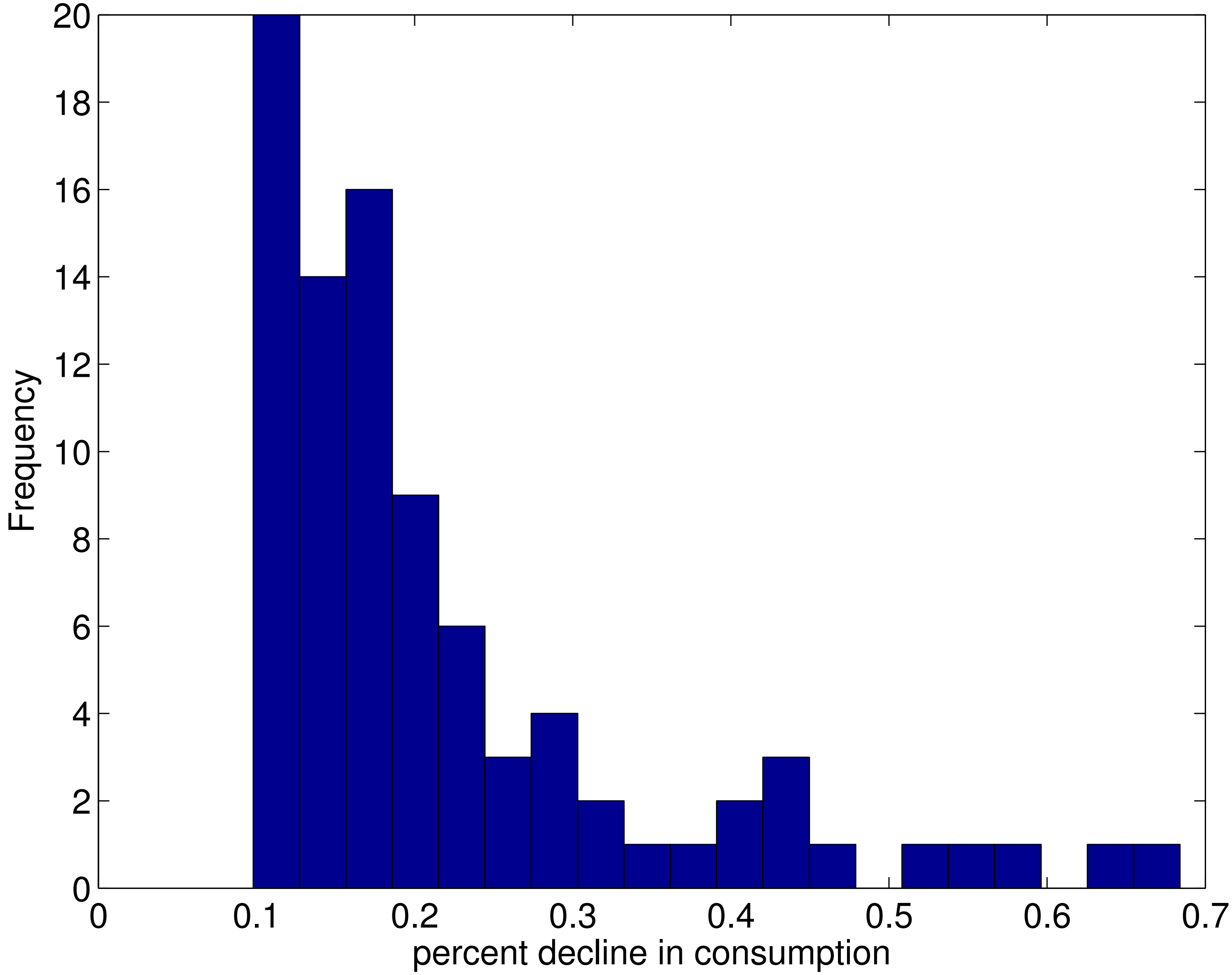


Parameter values

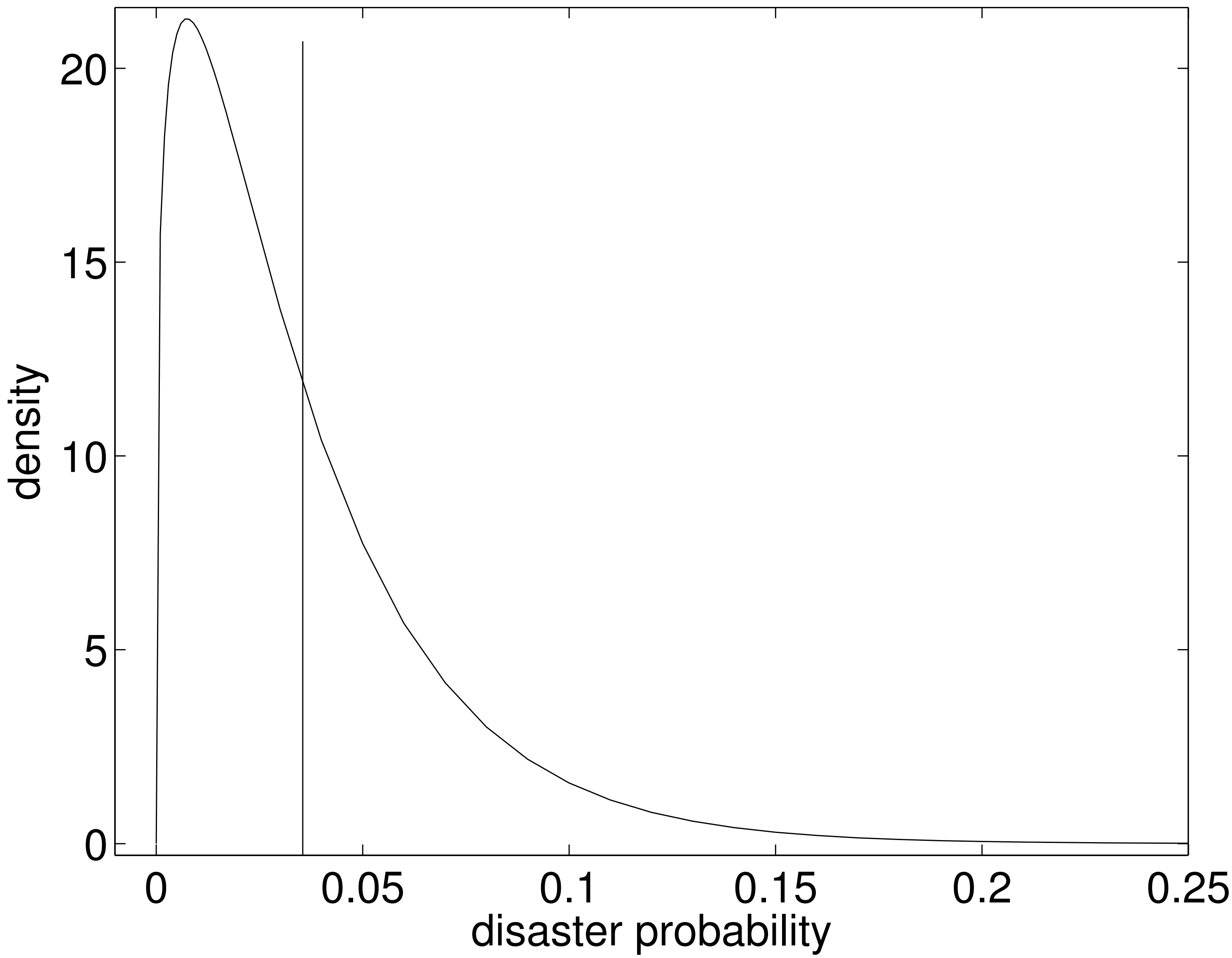
Panel A: Cash flow parameters	
Average growth in consumption (normal times) μ	0.0252
Volatility of consumption growth (normal times) σ	0.020
Leverage ϕ	2.6
Average probability of a rare disaster $\bar{\lambda}$	0.0355
Mean reversion κ	0.080
Volatility parameter σ_λ	0.067
$\sigma_\lambda E\left[\lambda^{1/2}\right]$	0.0114
Probability of default given disaster q	0.40
Panel B: Preference parameters	
Rate of time preference β	0.012
Relative risk aversion γ	3.0
Elasticity of intertemporal substitution ψ	1.0

Notes: Parameter values are in annual terms.

Distribution of GDP declines during disasters



Distribution of the disaster probability λ_t



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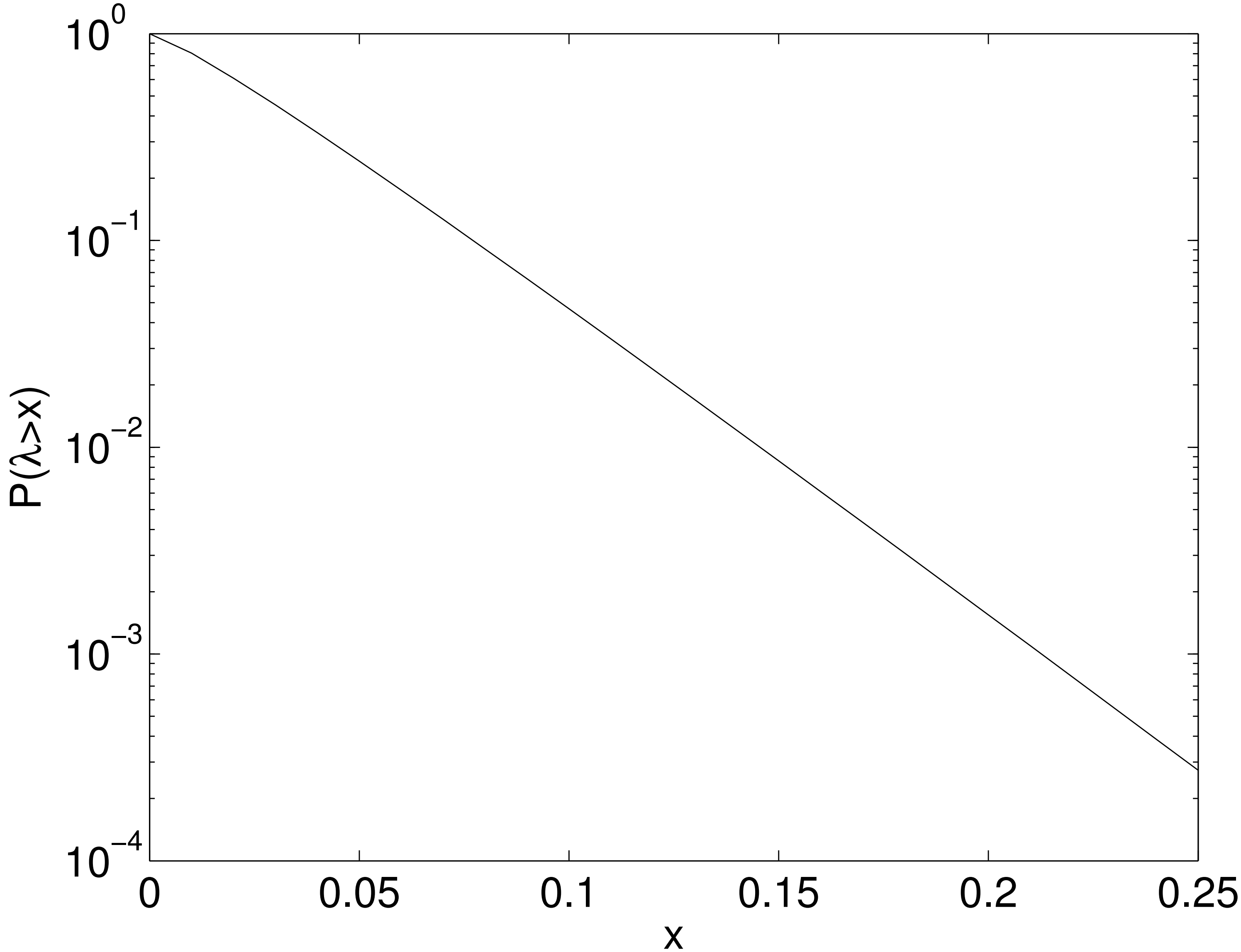


Population moments from simulated data and sample moments from the historical time series

	Model		U.S. Data
	Population	Conditional	
$E[R^b]$	0.99	1.36	1.34
$\sigma(R^b)$	3.79	2.00	2.66
$E[R^e - R^b]$	7.61	8.85	7.06
$\sigma(R^e)$	19.89	17.66	17.72
Sharpe Ratio	0.39	0.49	0.40
$\sigma(\Delta c)$	6.36	1.99	1.34
$\sigma(\Delta d)$	16.53	5.16	6.59

Notes: All moments are annual. R^b denotes the gross return on the government bond, R^e the gross equity return, Δc growth in log consumption and Δd growth in log dividends.

Long-horizon regressions: Excess returns

	Horizon in years					
	1	2	4	6	8	10
Panel A: Model – Population moments						
β_1	-0.11	-0.22	-0.40	-0.56	-0.69	-0.82
R^2	0.04	0.08	0.15	0.20	0.23	0.26
Panel B: Model – Conditional moments						
β_1	-0.16	-0.30	-0.56	-0.77	-0.95	-1.10
R^2	0.13	0.24	0.41	0.52	0.59	0.63
Panel B: U.S. Data						
β_1	-0.13	-0.23	-0.33	-0.48	-0.64	-0.86
t-stat	-2.62	-2.87	-3.64	-4.80	-5.82	-5.67
R^2	0.09	0.17	0.23	0.30	0.38	0.43

Note: The table reports statistics for regressions of excess returns on the lagged price-dividened ratio:

$$\sum_{i=1}^{h} \log(R^e_{t+i}) - \log(R^b_{t+i}) = \beta_0 + \beta_1(p_t - d_t) + \epsilon_t,$$

Long-horizon regressions: Consumption growth

	Horizon in years					
	1	2	4	6	8	10
Panel A: Model – Population moments						
β_1	0.02	0.04	0.07	0.10	0.12	0.13
R^2	0.01	0.02	0.04	0.05	0.06	0.06
Panel B: U.S. Data						
β_1	-0.001	-0.006	-0.009	-0.011	-0.016	-0.014
t-stat	-0.22	-0.85	-1.02	-1.15	-1.09	-0.79
R^2	0.0006	0.0137	0.0164	0.0180	0.0268	0.0162

Notes: The table reports coefficients β_1, R^2 statistics and, for the sample, Newey-West t-statistics for regressions

$$\sum_{i=1}^{h} \Delta c_{t+i} = \beta_0 + \beta_1(p_t - d_t) + \epsilon_t,$$

where Δc_{t+i} is log growth in aggregate consumption between periods $t+i-1$ and $t+i$ and $p_t - d_t$ is the log price-dividend ratio on the aggregated market. The conditional moments, namely the slope coefficient and the R^2 calculated over periods in the simulation without disasters, are equal to zero.

Inferring the disaster probability from observed prices

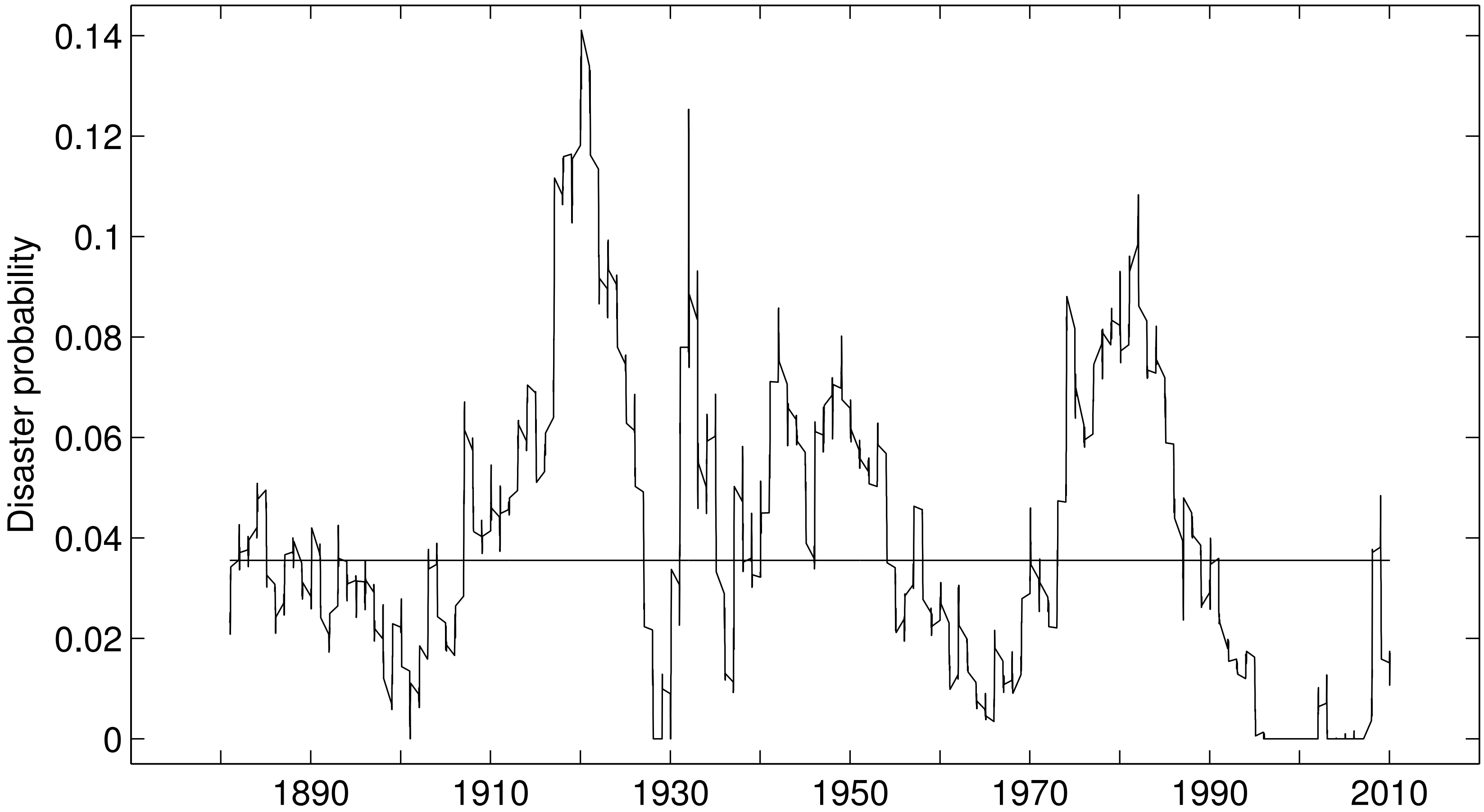
- The model implies that the price-dividend ratio is a function of the disaster probability.

- Given observations on the price-dividend ratio, I invert the function to back out values of λ_t.

- To avoid undue dependence on changes in dividend policy, I use the price divided by the previous ten years of earnings (Shiller, 1989; Graham and Dodd, 1934).

- If the resulting value of λ_t is negative, I set it to zero.

Inferred disaster probabilities



Conclusion

- An endowment model with a time-varying risk of a rare disaster can explain many features of the aggregate stock market.
- It can explain the high equity premium without assuming a high value of risk aversion for the representative investor.
- It can also explain the high level of stock market volatility and excess return predictability.
- The volatility of the government bill rate remains low due to a tradeoff between an increased desire to save and a greater risk of default.
- The model therefore offers a novel explanation of volatility in the aggregate stock market that is consistent with other macroeconomic data.
- The analytical solutions allow for straightforward computation and for potential extensions.