Internet Appendix for

"Free Cash Flow, Issuance Costs, and Stock Prices"*

This document is organized as follows. In Section I, we show that Proposition 1, which describes the dynamics of stock prices in the first-best benchmark, can be generalized to a broad class of dividend processes. Section II contains proofs omitted from the paper.

I. A Generalization of Proposition 1

Formulas (12) through (16) can be generalized to more general dividend processes. To see that, fix some nondecreasing process $\hat{L}$ such that $\hat{L}_0 = m$, which for simplicity we shall assume is continuous. To obtain the analogue of formula (13), we must ensure that the stock price exhibits no bubble, in the sense that it grows at an expected rate strictly lower than r, as in (12). This will be the case if the dividend process $\hat{L}$ grows at a fast enough rate. Specifically, we have the following result.

PROPOSITION IA.1: *Suppose that there are no issuance costs, that is, $f = 0$ and $p = 1$. Consider a continuous dividend process $\hat{L}$ such that*

$$\lim_{T\to\infty} \mathbf{E}\left[\exp\left(-\frac{1}{2}\left(\frac{\sigma r}{\mu}\right)^2 T + \frac{\sigma r}{\mu} W_T - \frac{r}{\mu}\hat{L}_T\right)\right] = 0. \tag{IA.1}$$

Then, at any time $t > 0$, the market capitalization of the firm is

$$\hat{N}_t \hat{S}_t = \frac{\mu}{r}, \tag{IA.2}$$

the instantaneous return on the shares issued by the firm satisfies

$$\frac{d\hat{S}_t + d\hat{L}_t/\hat{N}_t}{\hat{S}_t} = r dt + \frac{\sigma r}{\mu} dW_t, \tag{IA.3}$$

*Citation format: Décamps, Jean-Paul, Thomas Mariotti, Jean-Charles Rochet, and Stéphane Villeneuve, 2011, Internet Appendix for "Free Cash Flow, Issuance Costs, and Stock Prices," *Journal of Finance* 66, 1501–1544, http://www.afajof.org/supplements.asp. Please note: Wiley-Blackwell is not responsible for the content or functionality of any supporting information supplied by the authors. Any queries (other than missing material) should be directed to the authors of the article.

and the stock price is the present value of future dividends per share,

$$\hat{S}_t = \mathbf{E}\left[\int_t^\infty e^{-r(s-t)} \frac{d\hat{L}_s}{\hat{N}_s} \,|\, \mathcal{F}_t\right], \tag{IA.4}$$

P–*almost surely.*

Proof: From (4), the requirement that cash reserves be constant and equal to zero after time 0 yields the following analogue of (10):

$$\hat{I}_t = \hat{L}_t - \mu t - \sigma W_t \tag{IA.5}$$

for all $t \geq 0$. Using (11) along with (IA.5) then yields

$$\frac{d\hat{S}_t}{\hat{S}_t} = r dt + \frac{\sigma r}{\mu} dW_t - \frac{r}{\mu} d\hat{L}_t \tag{IA.6}$$

for all $t > 0$. The value of the firm at any time $t > 0$ is the present value of future cash flows, hence (IA.2). This allows us to rewrite (IA.6) as (IA.3). The following lemma holds.

LEMMA IA.1: *Given an initial condition* $\hat{S}_{0+} > 0$, *the stochastic differential equation (IA.6) has a unique strong solution, given by*

$$\hat{S}_t \equiv \hat{S}_{0+} \exp\left(\left[r - \frac{1}{2}\left(\frac{\sigma r}{\mu}\right)^2\right] t + \frac{\sigma r}{\mu} W_t - \frac{r}{\mu} \hat{L}_t\right) \tag{IA.7}$$

for all $t > 0$.

Proof: As $\hat{L}$ is continuous, any solution to (IA.6) is **P**–almost surely continuous. Using Itô's formula (Protter (1990, Chapter II, Theorem 32)), it is easy to check that the process $\hat{S} \equiv \{\hat{S}_t; t > 0\}$ defined by (IA.7) solves the stochastic differential equation (IA.6). Consider another solution $\tilde{S} \equiv \{\tilde{S}_t; t > 0\}$ to (IA.6) with the same initial condition $\hat{S}_{0+}$ as $\hat{S}$. Applying Itô's formula again, one can verify that, for each $t > 0$,

$$\mathbf{E}[(\hat{S}_t - \tilde{S}_t)^2] = \left(2r + \frac{\sigma^2 r^2}{2\mu^2}\right) \int_0^t \mathbf{E}[(\hat{S}_s - \tilde{S}_s)^2] \, ds - \frac{2r}{\mu} \mathbf{E}\left[\int_0^t (\hat{S}_s - \tilde{S}_s)^2 \, d\hat{L}_s\right]$$

$$\leq \left(2r + \frac{\sigma^2 r^2}{2\mu^2}\right) \int_0^t \mathbf{E}[(\hat{S}_s - \tilde{S}_s)^2]\, ds$$
$$\leq 0,$$

where the first inequality follows from the fact that $\hat{L}$ is a nondecreasing process, and the second from the first and Gronwall's lemma. Thus, one has $\hat{S}_t = \tilde{S}_t$, $\mathbf{P}$–almost surely, for all $t > 0$. As the processes $\hat{S}$ and $\tilde{S}$ are $\mathbf{P}$–almost surely continuous, the result follows (Karatzas and Shreve (1991, Chapter 1, Problem 1.5)). Q.E.D.

It should be noted that the initial condition for $\hat{S}$ in Lemma IA.1 is stipulated at time 0^+, that is, immediately after the special dividend m is distributed at time 0. Letting $\hat{N}_{0+} \equiv 1$ without loss of generality, (14) yields $\hat{S}_{0+} = \mu/r$. To conclude the proof, one only needs to check that (IA.4) holds. This requires the following lemma.

LEMMA IA.2: *Suppose that condition (IA.1) holds. Then, for each* $t \geq 0$,

$$\lim_{T\to\infty} \mathbf{E}\left[\exp\left(-\frac{1}{2}\left(\frac{\sigma r}{\mu}\right)^2 T + \frac{\sigma r}{\mu} W_T - \frac{r}{\mu}\hat{L}_T\right) | \mathcal{F}_t\right] = 0,$$

$\mathbf{P}$*–almost surely.*

Proof: Denote by $\{X_T; T \geq 0\}$ the random variables within the expectations in (IA.1), and fix for each $t \geq 0$ and $T \geq t$ a random variable $X_{T,t}$ in the equivalence class of $\mathbf{E}[X_T \,|\, \mathcal{F}_t]$. We first show that the random variables $X_{T,t}$, $T \geq t$, have a $\mathbf{P}$–almost surely well-defined limit as T goes to ∞. For each $t \geq 0$, define

$$Z_t \equiv \exp\left(-\frac{1}{2}\left(\frac{\sigma r}{\mu}\right)^2 t + \frac{\sigma r}{\mu} W_t\right).$$

The process $Z \equiv \{Z_t; t \geq 0\}$ is a martingale, and $\mathbf{E}[Z_t] = 1$ for all $t \geq 0$. Now suppose that $T_2 \geq T_1 \geq t$. Then one has

$$X_{T_2,t} = \mathbf{E}\left[Z_{T_2} \exp\left(-\frac{r}{\mu}\hat{L}_{T_2}\right) | \mathcal{F}_t\right]$$

$$
\begin{aligned}
&= \mathbf{E}\Big[\mathbf{E}\Big[Z_{T_2}\exp\Big(-\frac{r}{\mu}\,\hat{L}_{T_2}\Big)|\,\mathcal{F}_{T_1}\Big]|\,\mathcal{F}_t\Big] \\
&\leq \mathbf{E}\Big[\mathbf{E}[Z_{T_2}\,|\,\mathcal{F}_{T_1}]\,\exp\Big(-\frac{r}{\mu}\,\hat{L}_{T_1}\Big)|\,\mathcal{F}_t\Big] \\
&= \mathbf{E}\Big[Z_{T_1}\exp\Big(-\frac{r}{\mu}\,\hat{L}_{T_1}\Big)|\,\mathcal{F}_t\Big] \\
&= X_{T_1,t},
\end{aligned}
$$

P–almost surely, where the inequality follows from the fact that $\hat{L}$ is a nondecreasing process, and the third equality from the fact that Z is a martingale. Therefore, the random variables $X_{T,t}$, $T \geq t$, **P**–almost surely decrease as a function of T. As they are strictly positive, they have a **P**–almost surely well-defined limit as T goes to ∞, as claimed. We now show that this limit is **P**–almost surely zero, which concludes the proof. As the process $\hat{L}$ is nonnegative,

$$
X_{T,t} \leq \mathbf{E}[Z_T\,|\,\mathcal{F}_t] = Z_t,
$$

P–almost surely, where the equality follows again from the fact that Z is a martingale. Because $\mathbf{E}[Z_t] = 1$, the strictly positive random variables $X_{T,t}$, $T \geq t$, are uniformly bounded above by an integrable random variable. As they converge **P**–almost surely to a well-defined limit as T goes to ∞,

$$
\mathbf{E}\Big[\lim_{T\to\infty} X_{T,t}\Big] = \lim_{T\to\infty} \mathbf{E}[X_{T,t}] = \lim_{T\to\infty} \mathbf{E}[X_T] = 0,
$$

where the first equality follows from Lebesgue's dominated convergence theorem, and the last from (IA.1). As $\lim_{T\to\infty} X_{T,t}$ is a nonnegative random variable, the result follows. Q.E.D.

We are now ready to complete the proof of Proposition IA.1. Using (IA.2) and (IA.6), it is easy to verify that, for each $t \geq 0$ and $T \geq t$,

$$
e^{-rt}\hat{S}_t = \mathbf{E}[e^{-rT}\hat{S}_T\,|\,\mathcal{F}_t] + \mathbf{E}\Big[\int_t^T e^{-rs}\,\frac{d\hat{L}_s}{\hat{N}_s}\,|\,\mathcal{F}_t\Big], \tag{IA.8}
$$

P–almost surely. Lemmas IA.1 and IA.2 imply that the first term on the right-hand side of

(IA.8) goes to zero as T goes to ∞, $\mathbf{P}$–almost surely. Because $\hat{L}$ is a nondecreasing process and $\hat{N}$ remains strictly positive, letting T go to ∞ in (IA.8) yields (IA.4) by the monotone convergence theorem. Q.E.D.

II. Omitted Proofs

Proof of Lemma A.1: Because V_{m_1} is smooth over $[0, m_1)$, differentiating (28) and using the definition (20) of $\mathcal{L}$ yields $-\lambda V'_{m_1} + \mathcal{L} V'_{m_1} = 0$ over $[0, m_1)$. Using this along with (29) and (30), one obtains that $V'''_{m_1-}(m_1) = 2\lambda/\sigma^2 > 0$. As $V''_{m_1}(m_1) = 0$ and $V'_{m_1}(m_1) = 1$, it follows that $V''_{m_1} < 0$, and thus $V'_{m_1} > 1$ over some interval $(m_1 - \varepsilon, m_1)$, where $\varepsilon > 0$. Now suppose by way of contradiction that $V'_{m_1}(m) \leq 1$ for some $m \in [0, m_1 - \varepsilon]$, and let $\tilde{m} \equiv \sup\{m \in [0, m_1 - \varepsilon] \,|\, V'_{m_1}(m) \leq 1\}$. Then $V'_{m_1}(\tilde{m}) = 1$ and $V'_{m_1} > 1$ over $(\tilde{m}, m_1)$, so that $V_{m_1}(m_1) - V_{m_1}(m) > m_1 - m$ for all $m \in (\tilde{m}, m_1)$. As $V_{m_1}(m_1) = [(r-\lambda)m_1 + \mu]/r$, this implies that for any such m,

$$
\begin{aligned}
V''_{m_1}(m) &= \frac{2}{\sigma^2}\{rV_{m_1}(m) - [(r-\lambda)m + \mu)]V'_{m_1}(m)\} \\
&< \frac{2}{\sigma^2}\{r[m - m_1 + V_{m_1}(m_1)] - (r-\lambda)m - \mu\} \\
&= \frac{2}{\sigma^2}\lambda(m - m_1) \\
&< 0,
\end{aligned}
$$

which contradicts the fact that $V'_{m_1}(\tilde{m}) = V'_{m_1}(m_1) = 1$. Therefore, $V'_{m_1} > 1$ over $[0, m_1)$, from which it follows as above that $V''_{m_1} < 0$ over $[0, m_1)$. Q.E.D.

Proof of Lemma A.2: Consider the solutions H_0 and H_1 to the linear second-order differential equation $-rH + \mathcal{L}H = 0$ over $[0, \infty)$ characterized by the initial conditions $H_0(0) = 1$, $H'_0(0) = 0$, $H_1(0) = 0$, and $H'_1(0) = 1$. We first show that H'_0 and H'_1 are strictly positive over $(0, \infty)$. Consider H'_0. As $H_0(0) = 1$ and $H'_0(0) = 0$, one has $H''_0(0) = 2r/\sigma^2 > 0$, so that $H'_0 > 0$ over some interval $(0, \varepsilon)$, where $\varepsilon > 0$. Now suppose by way of contradiction that $\tilde{m} \equiv \inf\{m \geq \varepsilon \mid H'_0(m) \leq 0\} < \infty$. Then $H'_0(\tilde{m}) = 0$ and $H''_0(\tilde{m}) \leq 0$. Because $-rH_0 + \mathcal{L}H_0 = 0$, it follows that $H_0(\tilde{m}) \leq 0$, which is impossible as $H_0(0) = 1$ and H_0

is strictly increasing over $[0, \tilde{m}]$. Thus, $H_0' > 0$ over $(0, \infty)$, as claimed. The proof for H_1' is similar and is therefore omitted. Note that both H_0 and H_1 remain strictly positive over $(0, \infty)$. Next, let $W_{H_0,H_1} \equiv H_0 H_1' - H_1 H_0'$ be the Wronskian of H_0 and H_1. We have $W_{H_0,H_1}(0) = 1$ and

$$\begin{aligned} W_{H_0,H_1}'(m) &= H_0(m)H_1''(m) - H_1(m)H_0''(m) \\ &= \frac{2}{\sigma^2}\left(H_0(m)\{rH_1(m) - [(r-\lambda)m + \mu]H_1'(m)\}\right. \\ &\qquad \left. - H_1(m)\{rH_0(m) - [(r-\lambda)m + \mu]H_0'(m)\}\right) \\ &= -\frac{2[(r-\lambda)m + \mu]}{\sigma^2} W_{H_0,H_1}(m) \end{aligned}$$

for all $m \geq 0$, from which Abel's identity follows by integration:

$$W_{H_0,H_1}(m) = \exp\left(-\frac{(r-\lambda)m^2 + 2\mu m}{\sigma^2}\right) \tag{IA.9}$$

for all $m \geq 0$. Because $W_{H_0,H_1} > 0$, H_0 and H_1 are linearly independent. As a result, (H_0, H_1) is a basis of the two-dimensional space of solutions to the equation $-rH + \mathcal{L}H = 0$. It follows in particular that for each $m_1 > 0$, $V_{m_1} = V_{m_1}(0)H_0 + V_{m_1}'(0)H_1$ over $[0, m_1]$. Using the boundary conditions $V_{m_1}(m_1) = [(r-\lambda)m_1 + \mu]/r$ and $V_{m_1}'(m_1) = 1$, one can solve for $V_{m_1}(0)$ and $V_{m_1}'(0)$ as follows:

$$V_{m_1}(0) = \frac{H_1'(m_1)[(r-\lambda)m_1 + \mu]/r - H_1(m_1)}{W_{H_0,H_1}(m_1)}, \tag{IA.10}$$

$$V_{m_1}'(0) = \frac{H_0(m_1) - H_0'(m_1)[(r-\lambda)m_1 + \mu]/r}{W_{H_0,H_1}(m_1)}. \tag{IA.11}$$

Using the explicit expression (IA.9) for W_{H_0,H_1} along with the fact that H_0 and H_1 are solutions to $-rH + \mathcal{L}H = 0$, it is easy to verify from (IA.10) and (IA.11) that

$$\begin{aligned} \frac{dV_{m_1}(0)}{dm_1} &= -\frac{\lambda}{r}\exp\left(\frac{(r-\lambda)m_1^2 + 2\mu m_1}{\sigma^2}\right)H_1'(m_1), \\ \frac{d^2V_{m_1}(0)}{dm_1^2} &= -\frac{2\lambda}{\sigma^2}\exp\left(\frac{(r-\lambda)m_1^2 + 2\mu m_1}{\sigma^2}\right)H_1(m_1) \end{aligned}$$

and that

$$\frac{dV'_{m_1}(0)}{dm_1} = \frac{\lambda}{r}\exp\left(\frac{(r-\lambda)m_1^2+2\mu m_1}{\sigma^2}\right)H'_0(m_1).$$
$$\frac{d^2V'_{m_1}(0)}{dm_1^2} = \frac{2\lambda}{\sigma^2}\exp\left(\frac{(r-\lambda)m_1^2+2\mu m_1}{\sigma^2}\right)H_0(m_1).$$

The result then follows immediately from the fact that $\lambda > 0$ and that H_0, H'_0, H_1, and H'_1 are strictly positive over $\mathbb{R}_{++}$. Q.E.D.

Proof of Lemma A.3: Recall that (32) holds whenever $\hat{m}_1 > \tilde{m}_1$ and $V_{\hat{m}_1}(m_p(\hat{m}_1)) - p[m_p(\hat{m}_1)+f] > 0$. Equation (A.2) can be rewritten as $\varphi(\bar{m}_1) = 0$, where

$$\varphi(m_1) \equiv V_{m_1}(m_p(m_1)) - V_{m_1}(0) - p[m_p(m_1)+f].$$

The function φ is well-defined and continuous over $[\tilde{m}_1, \hat{m}_1]$, with $\varphi(\tilde{m}_1) = -pf < 0$ and $\varphi(\hat{m}_1) = V_{\hat{m}_1}(m_p(\hat{m}_1)) - p[m_p(\hat{m}_1)+f] > 0$ by condition (32). Thus, φ has at least a zero over $(\tilde{m}_1, \hat{m}_1)$. To prove that it is unique, we show that φ is strictly increasing over $(\tilde{m}_1, \hat{m}_1)$. Using the envelope theorem to evaluate the derivative of φ, this amounts to showing that

$$\frac{\partial W}{\partial m_1}(m_p(m_1), m_1) > \frac{\partial W}{\partial m_1}(0, m_1)$$

for all $m_1 \in (\tilde{m}_1, \hat{m}_1)$, where $W(m, m_1) \equiv V_{m_1}(m)$ for all $(m, m_1) \in [0,\infty) \times (\tilde{m}_1, \hat{m}_1)$. Because $m_p(m_1) \in (0, m_1)$ for all $m_1 \in (\tilde{m}_1, \hat{m}_1)$, all that needs to be established is that for any such m_1, $(\partial W/\partial m_1)(\cdot, m_1)$ is strictly increasing over $[0, m_1]$. From (28) through (30), it is straightforward to verify that $(\partial W/\partial m_1)(\cdot, m_1)$ is the unique solution to the following Cauchy problem over $[0, m_1]$:

$$-r\frac{\partial W}{\partial m_1}(m, m_1) + \mathcal{L}\frac{\partial W}{\partial m_1}(m, m_1) = 0; \quad 0 \leq m \leq m_1, \tag{IA.12}$$
$$\frac{\partial^2 W}{\partial m \partial m_1}(m_1, m_1) = 0, \tag{IA.13}$$
$$\frac{\partial^3 W}{\partial^2 m \partial m_1}(m_1, m_1) = -\frac{2\lambda}{\sigma^2}. \tag{IA.14}$$

We are interested in the sign of $(\partial^2 W/\partial m \partial m_1)(m, m_1)$ for $m \in [0, m_1)$. Because of (IA.13) and (IA.14), $(\partial^2 W/\partial m \partial m_1)(\cdot, m_1) > 0$ over some interval $(m_1 - \varepsilon, m_1)$, where $\varepsilon > 0$. Now suppose by way of contradiction that $(\partial^2 W/\partial m \partial m_1)(m, m_1) \leq 0$ for some $m \in [0, m_1 - \varepsilon]$, and let $\tilde{m} \equiv \sup\{m \in [0, m_1 - \varepsilon] \mid (\partial^2 W/\partial m \partial m_1)(m, m_1) \leq 0\}$. Then $(\partial^2 W/\partial m \partial m_1)(\tilde{m}, m_1) = 0$ and $(\partial^2 W/\partial m \partial m_1)(m, m_1) > 0$ for all $m \in (\tilde{m}, m_1)$, so that $(\partial W/\partial m_1)(m, m_1) < 0$ for all $m \in (\tilde{m}, m_1)$ as $(\partial W/\partial m_1)(m_1, m_1) = -\lambda/r < 0$ by (IA.12) through (IA.14). This implies that for any such m, one has

$$\frac{\partial^3 W}{\partial^2 m \partial m_1}(m, m_1) = \frac{2}{\sigma^2}\left\{r \frac{\partial W}{\partial m_1}(m, m_1) - [(r - \lambda)m + \mu] \frac{\partial^2 W}{\partial m \partial m_1}(m, m_1)\right\} < 0,$$

which is impossible as $(\partial^2 W/\partial m \partial m_1)(\tilde{m}, m_1) = (\partial^2 W/\partial m \partial m_1)(m_1, m_1) = 0$. Therefore, $(\partial^2 W/\partial m \partial m_1)(\cdot, m_1) > 0$ over $[0, m_1)$, and the result follows. Note for further reference that the above argument also establishes that $(\partial W/\partial m_1)(\cdot, m_1) < 0$ over $[0, m_1]$. Q.E.D.

Proof of Proposition A.1: We first establish the existence and uniqueness of V. As explained above, any solution V to (24) through (27) that is twice continuously differentiable over $(0, \infty)$ must coincide with some V_{m_1} over $[0, \infty)$. Because $V(0)$ is nonnegative by (25), one must have $m_1 \leq \hat{m}_1$. Suppose first that $\hat{m}_1 \leq \tilde{m}_1$, and that $m_1 < \hat{m}_1$. Then $V(0) = V_{m_1}(0) > 0$. But, as $m_1 < \tilde{m}_1$, one has $V'_+(0) = V'_{m_1}(0) < p$. It follows that the maximum of the mapping $m \mapsto V(m) - p(m + f)$ over $[f, \infty)$ is either attained at $-f$, for a value of zero, or at zero, for a value of $V(0) - pf$. In either case, this is inconsistent with condition (25). It follows that $m_1 = \hat{m}_1$, and thus V is given by (A.1). Suppose next that $\hat{m}_1 > \tilde{m}_1$. The above argument can be used to show that $m_1 > \tilde{m}_1$. Two cases must be distinguished. If $V_{\hat{m}_1}(m_p(\hat{m}_1)) - p[m_p(\hat{m}_1) + f] > 0$, then Lemma A.3 establishes the uniqueness of a value $\bar{m}_1$ of $m_1 \in (\tilde{m}_1, \hat{m}_1)$ consistent with condition (25). It follows that $m_1 = \bar{m}_1$, and thus V is given by (A.3). Suppose finally that $V_{\hat{m}_1}(m_p(\hat{m}_1)) - p[m_p(\hat{m}_1) + f] \leq 0$. Defining φ as in the proof of Lemma A.3 and using the fact that φ is strictly increasing over $(\tilde{m}_1, \hat{m}_1)$, we obtain that φ has no zero over $(\tilde{m}_1, \hat{m}_1)$. Thus, condition (25) cannot be satisfied for $m_1 \in (\tilde{m}_1, \hat{m}_1)$. It follows that the maximum of the mapping $m \mapsto V(m) - p(m + f)$ over

$[f, \infty)$ must be attained at $-f$, for a value of zero. The only choice of m_1 that is then consistent with (25) is $m_1 = \hat{m}_1$, and thus V is given by (A.1).

We now verify that our solution V to (24) through (27) satisfies the variational inequalities (17) through (19) over $(0, \infty)$. Inequality (17) follows from (27) and Lemma A.1, whereas inequality (19) follows from (26) and (27) along with the fact that $\lambda > 0$. As for (18), two cases must be distinguished. Suppose first that $\hat{m}_1 \leq \tilde{m}_1$, and hence $V'_+(0) \leq p$. For each $m \geq 0$, the mapping $m' \mapsto V(m' - f) - p(m' - m)$ is then strictly decreasing over $[m, \infty)$, and thus (18) holds as $V(m) \geq V(m - f)$ for any such m. Suppose next that $\hat{m}_1 > \tilde{m}_1$, and hence $V'_+(0) > p$. If $m \geq m_p(m_1) + f$, the same reasoning as above applies and (18) holds. If $m_p(m_1) + f > m \geq 0$, the maximum of the mapping $m' \mapsto V(m' - f) - p(m' - m)$ over $[m, \infty)$ is attained at $m_p(m_1) + f$, and we must therefore check that

$$V(m) - pm \geq V(m_p(m_1)) - p[m_p(m_1) + f] \tag{IA.15}$$

for any such m. The mapping $m \mapsto V(m) - pm$ is strictly increasing over $[0, m_p(m_1)]$ and strictly decreasing over $[m_p(m_1), m_p(m_1) + f]$. Thus, we only need to check that (IA.15) holds at $m = 0$ and at $m = m_p(m_1) + f$. The latter point is immediate. For the former, two cases must be distinguished. If (31) holds, then $m_1 = \hat{m}_1$ and (IA.15) holds at $m = 0$, because the right-hand side is at most zero, whereas the left-hand side is equal to zero as $V(0) = 0$. If (32) holds, then $m_1 = \bar{m}_1$ and (IA.15) holds as an equality at $m = 0$, because, by construction, $V(0) = V(m_p(\bar{m}_1)) - p[m_p(\bar{m}_1) + f]$. Q.E.D.

Proof of Lemma 1: We precisely show that one can find versions of the conditional expectations in (39) such that the resulting process S^* has $\mathbf{P}$–almost surely continuous paths. From (39), it follows that the stock price process S^* is such that, for each $t \geq 0$,

$$e^{-rt} S^*_t = \mathbf{E}\left[\int_0^\infty e^{-rs} \frac{dL^*_s}{N^*_s} \,|\, \mathcal{F}_t\right] - \int_0^t e^{-rs} \frac{dL^*_s}{N^*_s}, \tag{IA.16}$$

$\mathbf{P}$–almost surely. By choosing for each $t \geq 0$ a random variable Y_t in the equivalence class of $\mathbf{E}\left[\int_0^\infty e^{-rs} (1/N^*_s)\, dL^*_s \,|\, \mathcal{F}_t\right]$, one obtains an $\{\mathcal{F}_t; t \geq 0\}$–adapted martingale $Y \equiv \{Y_t; t \geq 0\}$.

As the filtration $\{\mathcal{F}_t; t \geq 0\}$ is complete and right-continuous, one can choose Y_t for all $t \geq 0$ in such a way that the martingale Y is right-continuous with left-hand limits (Karatzas and Shreve (1991, Chapter 1, Theorem 3.13)). Because $\{\mathcal{F}_t; t \geq 0\}$ is the $\mathbf{P}$–augmentation of the filtration generated by W, Y is in fact $\mathbf{P}$–almost surely continuous (Karatzas and Shreve (1991, Chapter 3, Problem 4.16)). To conclude the proof, observe that because the cumulative dividend process L^* is continuous, so is the second term on the right-hand side of (IA.16). Q.E.D.

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