Internet Appendix

for

"Correlation Risk and Optimal Portfolio Choice"*

THIS INTERNET APPENDIX includes five subappendices, identified by roman letters from A to E. Internet Appendix A reports proofs for the propositions stated in the published article. Internet Appendix B provides details about the estimation procedure that we use in the empirical application of Section II of the article. In particular, it derives in closed form the GMM moment restrictions. In Internet Appendix C we solve and discuss a model specification alternative to that considered in Section I of the article, namely a specification that comprises stochastic interest rates and constant risk premia. Internet Appendix D reports results for the discrete-time analog of the continous-time specification of the article. In this discrete-time setting, we also discuss the implications of short selling and VaR-type constraints on portfolio allocations. Finally, Internet Appendix E reports tables and graphs in support of the robustness checks and extensions provided in the article.

Propositions, lemmas, and equation numbers are prefixed with the letter that identifies the appendix. Numbers without prefix refer to Propositions, lemmas, or equations in the main text. Tables and figures of this appendix are labeled as "Table (Figure) IA.LX", where X denotes the number of the Table (Figure) and L is the letter that identifies the Appendix.

*Citation format: Buraschi Andrea, Paolo Porchia, and FabioTrojani, 2010, Internet Appendix to "Correlation Risk and Optimal Portfolio Choice," *Journal of Finance* 65, 393-420, http://www.afajof.org/supplements.asp.

A. Proofs

Proof of Proposition 1: The dynamics of the correlation process implied by the Wishart covariance matrix diffusion (4) is computed using Itô's Lemma. Let

$$\rho(t) = \frac{\Sigma_{12}(t)}{\sqrt{\Sigma_{11}(t)\Sigma_{22}(t)}} \tag{IA.A1}$$

be the instantaneous correlation between the returns of the first and second risky assets and denote by σ_{ij}, q_{ij}, and ω_{ij} the ij^{th} component of the volatility matrix $\Sigma^{1/2}$, the matrix Q, and the matrix $\Omega'\Omega = kQ'Q$ in equation (4), respectively. Applying Itô's Lemma to (IA.A1) and using the dynamics for Σ_{11}, Σ_{22}, and Σ_{12}, implied by (4), it follows that

$$\begin{aligned}
d\rho = &\left[\frac{\omega_{12}}{\sqrt{\Sigma_{11}\Sigma_{22}}} - \frac{\rho}{2\Sigma_{11}}\omega_{11} - \frac{\rho}{2\Sigma_{22}}\omega_{22} + \frac{\rho}{2}\left(\frac{q_{11}^2+q_{21}^2}{\Sigma_{11}} + \frac{q_{12}^2+q_{22}^2}{\Sigma_{22}}\right)\right.\\
&\left.+(\rho^2-2)\frac{q_{11}q_{12}+q_{21}q_{22}}{\sqrt{\Sigma_{11}\Sigma_{22}}} + (1-\rho^2)\frac{m_{21}\Sigma_{11}+m_{12}\Sigma_{22}}{\sqrt{\Sigma_{11}\Sigma_{22}}}\right] dt\\
&-\left[\frac{\rho}{2\Sigma_{11}\Sigma_{22}}\left(\Sigma_{22}\sigma_{11}q_{11} + \Sigma_{11}\sigma_{12}q_{12}\right) - \frac{\sigma_{12}q_{11}+\sigma_{11}q_{12}}{\sqrt{\Sigma_{11}\Sigma_{22}}}\right] dB_{11}\\
&-\left[\frac{\rho}{2\Sigma_{11}\Sigma_{22}}\left(\Sigma_{11}\sigma_{22}q_{12} + \Sigma_{22}\sigma_{21}q_{11}\right) - \frac{\sigma_{22}q_{11}+\sigma_{21}q_{12}}{\sqrt{\Sigma_{11}\Sigma_{22}}}\right] dB_{21}\\
&-\left[\frac{\rho}{2\Sigma_{11}\Sigma_{22}}\left(\Sigma_{22}\sigma_{11}q_{21} + \Sigma_{11}\sigma_{12}q_{22}\right) - \frac{\sigma_{11}q_{22}+\sigma_{12}q_{21}}{\sqrt{\Sigma_{11}\Sigma_{22}}}\right] dB_{12}\\
&-\left[\frac{\rho}{2\Sigma_{11}\Sigma_{22}}\left(\Sigma_{11}\sigma_{22}q_{22} + \Sigma_{22}\sigma_{21}q_{21}\right) - \frac{\sigma_{21}q_{22}+\sigma_{22}q_{21}}{\sqrt{\Sigma_{11}\Sigma_{22}}}\right] dB_{22}.
\end{aligned} \tag{IA.A2}$$

$B_{ij}(t)$, $i,j=1,2$, are the entries of the 2×2 matrix of Brownian motions in (4). Therefore, the instantaneous drift of the correlation process is a quadratic polynomial with state-dependent coefficients:

$$\mathbb{E}\left[d\rho(t)\middle|\,\mathcal{F}_t\right] = \left[E_1(t)\,\rho(t)^2 + E_2(t)\,\rho(t) + E_3(t)\right] dt, \tag{IA.A3}$$

where the coefficients $E_1(t)$, $E_2(t)$, and $E_3(t)$ are given by

$$E_1(t) = \frac{q_{11}q_{12}+q_{21}q_{22}}{\sqrt{\Sigma_{11}(t)\Sigma_{22}(t)}} - m_{21}\sqrt{\frac{\Sigma_{11}(t)}{\Sigma_{22}(t)}} - m_{12}\sqrt{\frac{\Sigma_{22}(t)}{\Sigma_{11}(t)}}, \tag{IA.A4}$$

$$E_2(t) = -\frac{\omega_{11}}{2\Sigma_{11}} - \frac{\omega_{22}}{2\Sigma_{22}} + \frac{1}{2}\left(\frac{q_{11}^2+q_{21}^2}{\Sigma_{11}(t)} + \frac{q_{12}^2+q_{22}^2}{\Sigma_{22}(t)}\right), \tag{IA.A5}$$

$$E_3(t) = \frac{\omega_{12}}{\sqrt{\Sigma_{11}\Sigma_{22}}} - 2\frac{q_{11}q_{12}+q_{21}q_{22}}{\sqrt{\Sigma_{11}(t)\Sigma_{22}(t)}} + m_{21}\sqrt{\frac{\Sigma_{11}(t)}{\Sigma_{22}(t)}} + m_{12}\sqrt{\frac{\Sigma_{22}(t)}{\Sigma_{11}(t)}}. \tag{IA.A6}$$

The instantaneous conditional variance of the correlation process is easily obtained from equation (IA.A2) and is a third-order polynomial with state dependent coefficients:

$$\mathbb{E}\left[d\rho(t)^2\middle|\,\mathcal{F}_t\right] = \left[\left(1-\rho^2(t)\right)\left(\frac{q_{11}^2+q_{21}^2}{\Sigma_{11}(t)} + \frac{q_{12}^2+q_{22}^2}{\Sigma_{22}(t)} - 2\rho(t)\frac{q_{11}q_{12}+q_{21}q_{22}}{\sqrt{\Sigma_{11}(t)\Sigma_{22}(t)}}\right)\right] dt.$$

This concludes the proof. □

Proof of Proposition 2: Since markets are incomplete, we follow He and Pearson (1991) and represent any market price of risk as the sum of two orthogonal components, one of which is spanned by the asset returns. Since Brownian motion W can be rewritten as $W = B\overline{\rho} + Z\sqrt{1-\overline{\rho}'\overline{\rho}}$, for a standard bivariate Brownian motion Z independent of B, we rewrite the innovation component of the opportunity set dynamics as $\Sigma^{1/2}[Z,B]L$, with $L=[\sqrt{1-\overline{\rho}'\overline{\rho}},\overline{\rho}_1,\overline{\rho}_2]'$. Let Θ_ν be the matrix-valued extension of Θ that prices the matrix of Brownian motions $\mathcal{B}=[Z,B]$. By definition of the market price of risk, we have

$$\Sigma^{1/2}\Theta_\nu L=\Sigma\lambda, \tag{IA.A7}$$

from which

$$\Theta_\nu=\Sigma^{1/2}\lambda L'+\Sigma^{1/2}\nu \tag{IA.A8}$$

for any 2×3 matrix valued process ν such that $\Sigma\nu L=0_{2\times 1}$. Since Σ is nonsingular, it follows that ν must be of the form $\nu=[-\overline{\nu}\frac{\rho}{\sqrt{1-\overline{\rho}'\overline{\rho}}},\overline{\nu}]$. $\overline{\nu}$ is a 2×2-matrix that prices the shocks that drive the variance-covariance matrix process.

Given Θ_ν, the associated martingale measure implies a process ξ_ν of stochastic discount factors, defined for $t\in[0,T]$ by[1]

$$\xi_\nu(t)=e^{-rt-tr\left(\int_0^t\Theta_\nu'(s)\,d\mathcal{B}+\frac{1}{2}\int_0^t\Theta_\nu'(s)\Theta_\nu(s)ds\right)}. \tag{IA.A9}$$

Our dynamic portfolio choice problem allows for an equivalent static representation by means of the following dual problem, as shown by He and Pearson (1991):

$$J(x,\Sigma_0)=\inf_\nu\sup_\pi\mathbb{E}\left[\frac{X(T)^{1-\gamma}-1}{1-\gamma}\right], \tag{IA.A10}$$

$$\text{s.t.}\quad \mathbb{E}\left[\xi_\nu(T)X(T)\right]\le x, \tag{IA.A11}$$

where $X(0)=x$. In what follows, we focus on the solution of problem (IA.A10) to (IA.A11). The optimality conditions for the innermost maximization are

$$X(T)=\left(\psi\xi_\nu(T)\right)^{-\frac{1}{\gamma}}, \tag{IA.A12}$$

where the Lagrange multiplier for the static budget constraint is

$$\psi=x^{-\gamma}\mathbb{E}\left[\xi_\nu(T)^{\frac{\gamma-1}{\gamma}}\right]^\gamma.$$

It then follows that

$$J(x,\Sigma_0)=x^{1-\gamma}\inf_\nu\frac{1}{1-\gamma}\mathbb{E}\left[\xi_\nu(T)^{\frac{\gamma-1}{\gamma}}\right]^\gamma-\frac{1}{1-\gamma}. \tag{IA.A13}$$

Using (IA.A9) and (IA.A13), one can see that the solution requires the computation of the expected value of the exponential of a stochastic integral. A simple change of measure reduces the problem to the calculation of the expectation of the exponential of a deterministic integral. Let P^γ be the probability measure defined by the following Radon-Nykodim derivative with respect to the physical measure P:

$$\frac{dP^\gamma}{dP}=e^{-tr\left(\frac{\gamma-1}{\gamma}\int_0^T\Theta_\nu'(s)\,d\mathcal{B}(s)+\frac{1}{2}\frac{(\gamma-1)^2}{\gamma^2}\int_0^T\Theta_\nu'(s)\Theta_\nu(s)ds\right)}. \tag{IA.A14}$$

We denote expectations under P^γ by $\mathbb{E}^\gamma[\cdot]$. Then the minimizer of (IA.A13) is the solution to the following problem:[2]

$$\begin{aligned}
\widehat{J}(0,\Sigma_0)&=\inf_\nu\mathbb{E}\left[\xi_\nu(T)^{\frac{\gamma-1}{\gamma}}\right]\\
&=\inf_\nu\mathbb{E}^\gamma\left[e^{-\frac{\gamma-1}{\gamma}rT+\frac{1-\gamma}{2\gamma^2}tr\left(\int_0^T\Theta_\nu'(s)\Theta_\nu(s)ds\right)}\right]\\
&=\inf_\nu\mathbb{E}^\gamma\left[e^{-\frac{\gamma-1}{\gamma}rT+\frac{1-\gamma}{2\gamma^2}tr\left(\int_0^T\Sigma(s)\left(\lambda\lambda'+\nu\nu'\right)ds\right)}\right]\\
&=\inf_\nu\mathbb{E}^\gamma\left[e^{-\frac{\gamma-1}{\gamma}rT+\frac{1-\gamma}{2\gamma^2}tr\left(\int_0^T\Sigma(s)\left(\lambda\lambda'+\overline{\nu}'\overline{\nu}(I_2+\frac{\overline{\rho}\overline{\rho}'}{1-\overline{\rho}'\overline{\rho}})\right)ds\right)}\right].
\end{aligned} \tag{IA.A15}$$

Notice that the expression in the exponential of the expectation in (IA.A15) is affine in Σ. By the Girsanov Theorem, under the measure P^γ the stochastic process $\mathcal{B}^\gamma=[Z^\gamma,B^\gamma]$, defined as

$$\mathcal{B}^\gamma(t)=\mathcal{B}(t)+\frac{\gamma-1}{\gamma}\int_0^t\Theta_\nu(s)ds,$$

is a 2×3 matrix of standard Brownian motions. Therefore, the process (4) is an affine process also under the new probability measure P^γ:

$$d\Sigma(t) = \left[\Omega\Omega' + \left(M - \frac{\gamma-1}{\gamma}Q'(\overline{\rho}\lambda' + \overline{\nu}')\right)\Sigma(t) + \Sigma(t)\left(M - \frac{\gamma-1}{\gamma}Q'(\overline{\rho}\lambda' + \overline{\nu}')\right)'\right]dt$$
$$+ \Sigma^{1/2}(t)dB^\gamma(t)Q + Q'dB(t)^{\gamma\prime}\Sigma^{1/2}(t). \tag{IA.A16}$$

Using the Feynman Kac formula, it is known that if the optimal ν and $\widehat{J}$ solve the probabilistic problem (IA.A15), then they must also be a solution to the following Hamilton-Jacobi-Bellman (HJB) equation:

$$0 = \frac{\partial \widehat{J}}{\partial t} + \inf_{\overline{\nu}}\left\{\mathcal{A}\widehat{J} + \widehat{J}\left[-\frac{\gamma-1}{\gamma}r + \frac{1-\gamma}{2\gamma^2}tr\left(\Sigma\left(\lambda\lambda' + \overline{\nu}'\overline{\nu}\left(I_2 + \frac{\overline{\rho}\overline{\rho}'}{1-\overline{\rho}'\overline{\rho}}\right)\right)\right)\right]\right\}, \tag{IA.A17}$$

subject to the terminal condition $\widehat{J}(T,\Sigma) = 1$, where $\mathcal{A}$ is the infinitesimal generator of the matrix-valued diffusion (IA.A16), which is given by

$$\mathcal{A} = tr\left(\left(\Omega\Omega' + \left(M - \frac{\gamma-1}{\gamma}Q'(\overline{\rho}\lambda' + \overline{\nu}')\right)\Sigma + \Sigma\left(M - \frac{\gamma-1}{\gamma}Q'(\overline{\rho}\lambda' + \overline{\nu}')\right)'\right)\mathcal{D}\right)$$
$$+ tr(2\Sigma\mathcal{D}Q'Q\mathcal{D}), \tag{IA.A18}$$

where

$$\mathcal{D} := \begin{pmatrix} \frac{\partial}{\partial\Sigma_{11}} & \frac{\partial}{\partial\Sigma_{12}} \\ \frac{\partial}{\partial\Sigma_{21}} & \frac{\partial}{\partial\Sigma_{22}} \end{pmatrix}. \tag{IA.A19}$$

The generator is affine in Σ. The optimality condition for the optimal control ν, implied by HJB equation (IA.A17), is

$$-\frac{1}{\gamma}\Sigma\overline{\nu}\left(I_2 + \frac{\overline{\rho}\overline{\rho}'}{1-\overline{\rho}'\overline{\rho}}\right) = \frac{\partial}{\partial\overline{\nu}}tr\left((Q'\overline{\nu}'\Sigma + \Sigma\overline{\nu}Q)\frac{\mathcal{D}\widehat{J}}{\widehat{J}}\right) = \frac{\partial}{\partial\overline{\nu}}tr\left(\frac{\mathcal{D}\widehat{J}}{\widehat{J}}Q'\overline{\nu}'\Sigma + \Sigma\overline{\nu}Q\frac{\mathcal{D}\widehat{J}}{\widehat{J}}\right).$$

Applying rules for the derivative of trace operators, the right-hand side can be written as $\Sigma\left(\frac{\mathcal{D}\widehat{J}}{\widehat{J}} + \frac{\mathcal{D}\widehat{J}'}{\widehat{J}}\right)Q'$. It follows that

$$\overline{\nu} = -\gamma\left(\frac{\mathcal{D}\widehat{J}}{\widehat{J}} + \frac{\mathcal{D}\widehat{J}'}{\widehat{J}}\right)Q'\left(I_2 + \frac{\overline{\rho}\overline{\rho}'}{1-\overline{\rho}'\overline{\rho}}\right)^{-1}. \tag{IA.A20}$$

Note that $\left(I_2 + \frac{\overline{\rho}\overline{\rho}'}{1-\overline{\rho}'\overline{\rho}}\right)^{-1} = I_2 - \overline{\rho}\overline{\rho}'$. Substituting the expression for ν in equation (IA.A18), we obtain the generator

$$\mathcal{A} = tr\left(\left(\Omega\Omega' + \left(M - \frac{\gamma-1}{\gamma}Q'\overline{\rho}\lambda'\right)\Sigma + \Sigma\left(M - \frac{\gamma-1}{\gamma}Q'\overline{\rho}\lambda'\right)'\right)\mathcal{D} + 2\Sigma\mathcal{D}Q'Q\mathcal{D}\right)$$
$$+ (\gamma-1)\widehat{J}\,tr\left((I_2 - \overline{\rho}\overline{\rho}')\left(Q'Q\left(\frac{\mathcal{D}\widehat{J}}{\widehat{J}} + \frac{\mathcal{D}\widehat{J}'}{\widehat{J}}\right)\Sigma + \Sigma\left(\frac{\mathcal{D}\widehat{J}}{\widehat{J}} + \frac{\mathcal{D}\widehat{J}'}{\widehat{J}}\right)Q'Q\right)\mathcal{D}\right)$$
$$= tr\left(\left(\Omega\Omega' + \left(M - \frac{\gamma-1}{\gamma}Q'\overline{\rho}\lambda'\right)\Sigma + \Sigma\left(M - \frac{\gamma-1}{\gamma}Q'\overline{\rho}\lambda'\right)'\right)\mathcal{D} + 2\Sigma\mathcal{D}Q'Q\mathcal{D}\right)$$
$$- (1-\gamma)\widehat{J}\,tr\left((I_2 - \overline{\rho}\overline{\rho}')\Sigma\left(\frac{\mathcal{D}\widehat{J}}{\widehat{J}} + \frac{\mathcal{D}\widehat{J}'}{\widehat{J}}\right)Q'Q\left(\frac{\mathcal{D}}{\widehat{J}} + \frac{\mathcal{D}'}{\widehat{J}}\right)'\right).$$

Substitution of the last expression for $\mathcal{A}$ into the HJB equation (IA.A17) yields the following partial differential equation for $\widehat{J}$:

$$-\frac{\partial\widehat{J}}{\partial t} = tr\left(\left(\Omega\Omega' + \left(M - \frac{\gamma-1}{\gamma}Q'\overline{\rho}\lambda'\right)\Sigma + \Sigma\left(M - \frac{\gamma-1}{\gamma}Q'\overline{\rho}\lambda'\right)'\right)\mathcal{D} + 2\Sigma\mathcal{D}Q'Q\mathcal{D}\right)\widehat{J}$$
$$+ \frac{\gamma-1}{\gamma}\widehat{J}\left(-r - \frac{tr(\Sigma\lambda\lambda')}{2\gamma}\right) - \frac{1-\gamma}{2}\widehat{J}\,tr\left((I_2 - \overline{\rho}\overline{\rho}')\Sigma\left(\frac{\mathcal{D}\widehat{J}}{\widehat{J}} + \frac{\mathcal{D}\widehat{J}'}{\widehat{J}}\right)Q'Q\left(\frac{\mathcal{D}\widehat{J}}{\widehat{J}} + \frac{\mathcal{D}\widehat{J}'}{\widehat{J}}\right)'\right),$$

subject to the boundary condition $\widehat{J}(\Sigma, T) = 1$. The affine structure of this problem suggests an exponentially affine functional form for its solution,

$$\widehat{J}(t, \Sigma) = \exp(B(t,T) + tr(A(t,T)\Sigma),$$

for some state-independent coefficients $B(t,T)$ and $A(t,T)$. After inserting this functional form into the differential equation for $\widehat{J}$, the guess can be easily verified. The coefficients B and A are the solutions of the following system of Riccati equations:

$$-\frac{dB}{dt} = tr(A\Omega\Omega') - \frac{\gamma-1}{\gamma}r,$$

$$-tr\left(\frac{dA}{dt}\Sigma\right) = tr\left(\Gamma' A\Sigma + A\Gamma\Sigma + 2AQ'QA\Sigma - \frac{1-\gamma}{2}(A'+A)Q'(I_2 - \overline{\rho\rho}')Q(A'+A)\Sigma + C\Sigma\right),$$

with terminal conditions $B(T,T) = 0$ and $A(T,T) = 0_{2\times 2}$, where

$$\Gamma = M - \frac{\gamma-1}{\gamma}Q'\overline{\rho}\lambda' \quad \text{(IA.A21)}$$

$$C = \frac{1-\gamma}{2\gamma^2}\lambda\lambda'. \quad \text{(IA.A22)}$$

For a symmetric matrix function A, the second differential equation implies the following matrix Riccati equation:

$$0_{2\times 2} = \frac{dA}{dt} + \Gamma' A + A\Gamma + 2A\Lambda A + C, \quad \text{(IA.A23)}$$

where

$$\Lambda = Q'(I_2\gamma + (1-\gamma)\overline{\rho\rho}')Q. \quad \text{(IA.A24)}$$

This is the system of matrix Riccati equations in the statement of Proposition 2. These differential equations are completely integrable, so that closed-form expressions for $\widehat{J}$ (and hence for J) can be computed. For convenience, we consider coefficients A and B parameterized by $\tau = T - t$. This change of variable implies the following simple modification of the above system of equations:

$$\frac{dB}{d\tau} = tr(A\Omega\Omega') - \frac{\gamma-1}{\gamma}r, \quad \text{(IA.A25)}$$

$$\frac{dA}{d\tau} = \Gamma' A + A\Gamma + 2A\Lambda A + C, \quad \text{(IA.A26)}$$

subject to initial conditions $A(0) = 0_{2\times 2}$ and $B(0) = 0$. Given a solution for A, function B is obtained by integration:

$$B(\tau) = tr\left(\int_0^\tau A(s)\Omega\,\Omega'\,ds\right) - \frac{\gamma-1}{\gamma}r\,\tau.$$

To solve equation (IA.A26), we use Radon's Lemma. Let us represent the function $A(\tau)$ as

$$A(\tau) = H(\tau)^{-1}K(\tau), \quad \text{(IA.A27)}$$

where $H(\tau)$ and $K(\tau)$ are square matrices, with $H(\tau)$ invertible. Pre-multiplying (IA.A26) by $H(\tau)$, we obtain

$$H\frac{dA}{d\tau} = H\Gamma' A + HA\Gamma + 2HA\Lambda A + HC. \quad \text{(IA.A28)}$$

Where no confusion may arise, we suppress the argument τ for brevity. On the other hand, in light of (IA.A27), differentiation of

$$HA = K \quad \text{(IA.A29)}$$

results in

$$H\frac{dA}{d\tau} = \frac{d}{d\tau}(HA) - \frac{dH}{d\tau}A, \quad \text{(IA.A30)}$$

and

$$\frac{d}{d\tau}(HA) = \frac{dK}{d\tau}. \tag{IA.A31}$$

Substituting (IA.A29), (IA.A30), and (IA.A31) into (IA.A28) we get

$$\frac{dK}{d\tau} - \frac{dH}{d\tau}A = H\Gamma' A + K\Gamma + 2K\Lambda A + HC.$$

After collecting coefficients of A, we conclude that the last equation is equivalent to the following matrix system of ODEs:

$$\frac{dK}{d\tau} = K\Gamma + HC, \tag{IA.A32}$$

$$\frac{dH}{d\tau} = -2K\Lambda - H\Gamma', \tag{IA.A33}$$

or

$$\frac{d}{d\tau}(K \quad H) = (K \quad H)\begin{pmatrix} \Gamma & -2\Lambda \\ C & -\Gamma' \end{pmatrix}.$$

The above ODE can be solved by exponentiation:

$$\begin{aligned}
(\quad K(\tau) \quad H(\tau) \quad) &= (\quad K(0) \quad H(0) \quad) \exp\left[\tau \begin{pmatrix} \Gamma & -2\Lambda \\ C & -\Gamma' \end{pmatrix}\right] \\
&= (\quad A(0) \quad I_2 \quad) \exp\left[\tau \begin{pmatrix} \Gamma & -2\Lambda \\ C & -\Gamma' \end{pmatrix}\right] \\
&= (\quad A(0)F_{11}(\tau) + F_{21}(\tau) \quad A(0)F_{12}(\tau) + F_{22}(\tau) \quad) \\
&= (\quad F_{21}(\tau) \quad F_{22}(\tau) \quad).
\end{aligned}$$

We conclude from equation (IA.A27) that the solution to (IA.A26) is given by

$$A(\tau) = F_{22}(\tau)^{-1}F_{21}(\tau). \tag{IA.A34}$$

This concludes the proof. □

Proof of Proposition 3: To recover the optimal portfolio policy, we have, from the proof of Proposition 2,

$$X^*(t) =: \frac{1}{\xi_{\nu^*}(t)}\mathbb{E}[\xi_{\nu^*}(T)X^*(T)\,|\mathcal{F}_t] = \psi^{-\frac{1}{\gamma}}\xi_{\nu^*}(t)^{-\frac{1}{\gamma}}\widehat{J}(t, \Sigma(t)). \tag{IA.A35}$$

For the Wishart dynamics (4), Itô's Lemma applied to both sides of (IA.A35) gives, for every state Σ,

$$X^*(t)\, tr\left(\begin{bmatrix}\pi_1 & \pi_2\end{bmatrix}\Sigma^{1/2}d\mathcal{B}L\right) = X^*(t)\, tr\left(\frac{1}{\gamma}\Theta'_{\nu^*}d\mathcal{B} + \frac{\mathcal{D}\widehat{J}'}{\widehat{J}}\left(\Sigma^{1/2}d\mathcal{B}UQ + Q'U'd\mathcal{B}'\Sigma^{1/2}\right)\right). \tag{IA.A36}$$

where matrix U is given by:

$$U = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

This implies

$$L\begin{bmatrix}\pi_1 & \pi_2\end{bmatrix}\Sigma^{1/2} = \frac{1}{\gamma}\,(L\lambda' + \nu')\,\Sigma^{1/2} + 2UQA\Sigma^{1/2}.$$

Pre-multiplying both sides by L', post-multiplying them by $\Sigma^{1/2}$, and recalling that $L'\nu'\Sigma = 0_{1\times 2}$, we conclude that portfolio weight $\pi = (\pi_1, \pi_2)'$ is

$$\pi = \frac{\lambda}{\gamma} + 2AQ'\overline{\rho} = \frac{1}{\gamma}\begin{bmatrix}\lambda_1 \\ \lambda_2\end{bmatrix} + 2\begin{bmatrix}(q_{11}\overline{\rho}_1 + q_{21}\overline{\rho}_2)A_{11} + (q_{12}\overline{\rho}_1 + q_{22}\overline{\rho}_2)A_{12} \\ (q_{12}\overline{\rho}_1 + q_{22}\overline{\rho}_2)A_{22} + (q_{11}\overline{\rho}_1 + q_{21}\overline{\rho}_2)A_{12}\end{bmatrix}. \tag{IA.A37}$$

This concludes the proof of the proposition. □

Proof of Proposition 4: We apply the following lemma, similar to a result in Buraschi, Cieslak and Trojani (2007), to which we refer for a proof.

LEMMA IA.A1: *Consider the solution $A(\tau)$ of matrix Riccati equation (IA.A26). If matrix C is negative semidefinite, then $A(\tau)$ is negative semidefinite and monotonically decreasing for any τ, that is, $A(\tau_2)-A(\tau_1)$ is a negative semidefinite matrix for any $\tau_2 > \tau_1$.*

Since $C = (1-\gamma)/(2\gamma^2)\lambda\lambda'$, if $\gamma > 1$ then C is negative semidefinite. From Lemma IA.A1, $A(\tau)$ is also negative semidefinite. It follows that $A_{11}(\tau) \leq 0$ and $A_{22}(\tau) \leq 0$. Inequality $|A_{12}| \leq |A_{11} + A_{22}|/2$ follows from the properties of negative semidefinite matrices. Now consider a neighborhood of $\tau = 0$ of arbitrary small length ϵ. By the fundamental theorem of calculus, we have

$$A(\epsilon) = A(0) + \left.\frac{dA(\tau)}{d\tau}\right|_{\tau=0} \epsilon + o(\epsilon). \tag{IA.A38}$$

But $A(0) = 0$ and $\left.\frac{dA(\tau)}{d\tau}\right|_{\tau=0} = C$. If λ_1 and λ_2 agree in sign and $\gamma > 1$, then $C_{12} < 0$ and $A_{12}(\epsilon) < 0$. If, in addition, $|\lambda_1| > |\lambda_2|$, we have $\lambda_1^2 > \lambda_1\lambda_2 > \lambda_2^2$, that is, $|C_{11}| > |C_{12}| > |C_{22}|$. We conclude from (IA.A38) that $|A_{11}| > |A_{12}| > |A_{22}|$. This concludes the proof of the proposition. $\square$

B. Moment Restrictions for the GMM Estimation

This Appendix provides detailed expressions for the moment conditions used in the GMM estimation of our model. The following computations make use of the closed-form expressions for the moments of the Wishart process, which can be found, for example, in the Appendix of Buraschi, Cieslak, and Trojani (2007).

Let τ denote data sampling frequency. We have $\tau = 5/250$ for weekly data and $\tau = 22/250$ for monthly data.

1) Unconditional risk premia of log returns.
The conditional risk premia of asset i's logarithmic returns, at frequency τ, $i = 1, 2$, are given by

$$\mathbb{E}_t\left[\log S^i(t+\tau) - \log S^i(t)\right] - \int_t^{t+\tau} r ds = \mathbb{E}_t\left[\int_t^{t+\tau} e_i'\Sigma(s)(\lambda - \frac{1}{2}e_i)ds\right]. \tag{IA.B1}$$

The unconditional risk premia are thus

$$M_1 = \left(\mathbb{E}[\Sigma(t)]\lambda - \frac{1}{2}\left[\begin{array}{c} e_1'\mathbb{E}[\Sigma(t)]e_1 \\ e_2'\mathbb{E}[\Sigma(t)]e_2 \end{array}\right]\right)\tau. \tag{IA.B2}$$

2) Unconditional mean of the realized variance-covariance matrix of log returns.

$$M_2 = \text{vech}\left(\mathbb{E}[\Sigma(t)]\right)\tau, \tag{IA.B3}$$

where vech(X) denotes the lower triangular vectorization of a square matrix X.

3) Unconditional second moment of the realized variance-covariance matrix of log-returns.

$$\begin{aligned}\mathbb{E}\left[\left(\int_t^{t+\tau} \text{vec}\left(\Sigma(s)\right)ds\right)\left(\int_t^{t+\tau} \text{vec}\left(\Sigma(s)\right)ds\right)'\right] &= 2\int_0^\tau dr_2\int_0^{r_2} dr_1\mathbb{E}\left[\text{vec}(\Sigma(r_1))\text{vec}(\Sigma(r_2))'\right] \quad \text{(IA.B4)}\\ &= 2\int_0^\tau dr_2\int_0^{r_2} dr_1\mathbb{E}\left[\text{vec}(\Sigma(r_1))\mathbb{E}_{r_1}\left[\text{vec}(\Sigma(r_2))'\right]\right]\end{aligned}$$

$$\begin{aligned}= 2\Bigg(&\mathbb{E}\left[\text{vec}(\Sigma(r_1))\text{vec}(\Sigma(r_1))'\right]\int_0^\tau dr_2\int_0^{r_2} dr_1(\exp(M'(r_2-r_1))\otimes\exp(M'(r_2-r_1)))\\ &+\mathbb{E}[\text{vec}(\Sigma(r_1))]\int_0^\tau dr_2\int_0^{r_2} dr_1\text{vec}\left(\int_0^{r_2-r_1}\exp(sM)kQ'Q\exp(sM')\right)' ds\Bigg).\end{aligned}$$

Therefore,

$$M_3 = \text{vech}\left(\mathbb{E}\left[\left(\int_t^{t+\tau}\text{vec}\left(\Sigma(s)\right)ds\right)\left(\int_t^{t+\tau}\text{vec}\left(\Sigma(s)\right)ds\right)'\right]\right).$$

4) Unconditional covariance between assets' simple excess returns and the variance-covariance matrix of log returns. For asset i, $i = 1, 2$, and $s > t$, we have

$$\begin{aligned}&\lim_{t\to\infty}\mathbb{E}_t\left[\exp\left(\int_s^{s+\tau} e_i\Sigma(u)dW(u) + \int_s^{s+\tau} e_i\Sigma(u)\lambda du - \frac{1}{2}\int_s^{s+\tau} e_i\Sigma(u)e_i' du\right)\otimes\int_s^{s+\tau}\Sigma(u)du\right] = \\ &\exp\left(A_l(\tau)+\widetilde{A}_l(\infty)\right)\left(\int_0^\tau \exp(\widetilde{M}(u)u)\mathbb{E}\left[\Sigma(t)\right]\exp(\widetilde{M}(u)'u)du + \int_0^\tau\int_0^s \exp(\widetilde{M}(u)u)kQ'Q\exp(\widetilde{M}(u)'u)du\, ds\right),\end{aligned} \tag{IA.B5}$$

where

$$\begin{aligned}\widetilde{M}(\tau) &= M + Q'\rho e_i' + Q'QB_l(\tau)\\ A_l(\tau) &= k\textstyle\int_0^\tau tr\left(B_l(s)Q'Q\right)ds\\ \widetilde{A}_l(\infty) &= k\textstyle\int_0^\infty tr\left(\widetilde{B}_l(s)Q'Q\right)ds\\ B_l(t) &= B_{22}(t)^{-1}B_{21}(t)\\ \widetilde{B}_l(t) &= (B_l(t)\widetilde{B}_{12}(t)+\widetilde{B}_{22}(t))^{-1}B_l(t)\widetilde{B}_{11}(t)\end{aligned}$$

and

$$\begin{pmatrix} B_{11}(t) & B_{12}(t) \\ B_{21}(t) & B_{22}(t) \end{pmatrix} = \exp\left[t\begin{pmatrix} M + Q'\rho e_i' & -2Q'Q \\ \lambda e_i' & -(M + Q'\rho e_i')' \end{pmatrix}\right]$$

$$\begin{pmatrix} \widetilde{B}_{11}(t) & \widetilde{B}_{12}(t) \\ \widetilde{B}_{21}(t) & \widetilde{B}_{22}(t) \end{pmatrix} = \exp\left[t\begin{pmatrix} M & -2Q'Q \\ 0 & -M' \end{pmatrix}\right].$$

The last set of moment conditions is therefore given by

$$\begin{aligned} &M_{3+i} = \mathrm{vech}\left(\exp\left(A_l(\tau) + \widetilde{A}_l(\infty)\right) \times \right. \\ &\left.\left(\int_0^\tau \exp(\widetilde{M}(u)u)\mathbb{E}\left[\Sigma(t)\right]\exp(\widetilde{M}(u)'u)du + \int_0^\tau \int_0^s \exp(\widetilde{M}(u)u)kQ'Q\exp(\widetilde{M}(u)'u)du\, ds\right)\right) \end{aligned} \tag{IA.B6}$$

for $i = 1, 2$. Summarizing, the vector-valued function $\mu^\tau(M, Q, \lambda, \rho, k)$ of theoretical moment conditions, for sampling frequency τ, is given by

$$\mu^\tau(M, Q, \lambda, \rho, k) = \begin{bmatrix} M_1 \\ M_2 \\ M_3 \\ M_4 \\ M_5 \end{bmatrix}.$$

In our GMM estimation, this is compared to its empirical counterpart $\hat{\mu}^\tau$ based on historical returns, volatilities, and covariances.

C. Constant Risk Premia and Stochastic Interest Rate

A direct way to study pure variance-covariance hedging demands is by assuming a constant risk premium. For analytical purposes, this comes at the cost of specifying a Wishart state process for the precision matrix Σ^{-1}, which implies a less transparent interpretation of some model parameters. This can be achieved even in a setting with a stochastic interest rate, where the interest rate can also depend on some of the risk factors driving the covariance matrix of asset returns.[3]

ASSUMPTION IA.C1: *Let the process Y satisfy the following Wishart dynamics:*

$$dY(t) = [\Omega\Omega' + MY(t) + Y(t)M']dt + Y^{1/2}(t)dBQ + Q'dB'Y^{1/2}(t), \tag{IA.C1}$$

where matrices Ω, M, and Q are now of dimension 3×3 and where B is a 3×3 matrix of independent standard Brownian motions. We model Σ^{-1} as a projection of matrix Y:

$$\Sigma^{-1} = SYS',$$

where the 2×3 matrix S is such that $SS' = id_{2\times 2}$.[4] *The stochastic riskless rate $r(t)$ is defined by*

$$r(t) = r_0 + tr(Y(t)D), \tag{IA.C2}$$

where $r_0 > 0$ and D is a 3×3 matrix.

Notice that the nonnegativity of $r(t)$ can be easily ensured simply by assuming that matrix D is positive definite. Since $\Sigma^{-1} = SYS'$, we define $\Sigma^{-1/2}$ as the 2×3 matrix $SY^{-1/2}$. Since $\Sigma^{-1/2}\Sigma^{1/2\prime} = id_{2\times 2}$, it is natural to define $\Sigma^{1/2}$ as the 2×3 matrix $SY^{1/2}$. We introduce the following process for asset returns:

$$dS(t) = I_S\left[\begin{pmatrix} r(t)+\mu_1^e \\ r(t)+\mu_2^e \end{pmatrix} dt + \Sigma^{1/2}(t)dW(t)\right], \tag{IA.C3}$$

where the excess return vector $\mu^e = (\mu_1^e, \mu_2^e)' \in \mathbb{R}^2$ is constant and $r(t)$ is given by equation (IA.C2). To model leverage effects, we define the standard Brownian motion W as

$$W(t) = \sqrt{1-\overline{\rho}'\overline{\rho}}Z(t) + B(t)\overline{\rho}, \tag{IA.C4}$$

where Z is a three-dimensional standard Brownian motion independent of B and $\overline{\rho} = (\overline{\rho}_1, \overline{\rho}_2, \overline{\rho}_3)'$ is a vector of correlation parameters such that $\overline{\rho}_i \in [-1, 1]$ and $\overline{\rho}'\overline{\rho} \le 1$.

This setting is effectively a six-factor model with some interest rate risk factors that might be linked to the covariance matrix of stock returns, depending on the form of the matrix D in equation (IA.C2). The squared Sharpe ratio in this model is affine in Y. Therefore, we can solve in closed form the dynamic portfolio choice problem in this extended dynamic setting as well.

PROPOSITION IA.C1: *The solution to the portfolio problem for the return dynamics (IA.C1) to (IA.C3) and under a stochastic interest rate (IA.C2) is*

$$J(X_0, Y_0) = \frac{X_0^{1-\gamma}\widehat{J}(0, Y_0)^{\gamma} - 1}{1-\gamma},$$

where

$$\widehat{J}(t, Y) = \exp\left(B(t,T) + tr\left(A(t,T)Y\right)\right),$$

with $B(t,T)$ and the symmetric matrix-valued function $A(t,T)$ solving in closed form the following system of matrix Riccati differential equations:

$$-\frac{dB}{dt} = -\frac{\gamma-1}{\gamma}r_0 + tr(A\Omega\Omega'), \tag{IA.C5}$$

$$-\frac{dA}{dt} = \Gamma'A + A'\Gamma + 2A'\Lambda A + C, \tag{IA.C6}$$

subject to $B(T,T)=0$ and $A(T,T)=0$. In these equations, the coefficients Γ, Λ, and C are given by

$$\begin{aligned}\Gamma &= M - \frac{\gamma}{\gamma-1} Q'\overline{\rho}\mu^{e\prime}S\\ \Lambda &= Q'(\gamma I_3 + (1-\gamma)\overline{\rho}\overline{\rho}')Q\\ C &= \frac{1-\gamma}{2\gamma^2} S'\mu^e\mu^{e\prime}S - \frac{\gamma-1}{\gamma}D.\end{aligned}$$

Finally, the optimal policy for this portfolio problem reads

$$\pi = \frac{1}{\gamma}\Sigma^{-1}\mu^e + 2\Sigma^{-1}SAQ'\overline{\rho}\,. \tag{IA.C7}$$

Proof. Analogous to the proof of Proposition 2, we rewrite the innovation component of the opportunity set dynamics as $\Sigma^{1/2}[Z,B]L$, with $L=[\sqrt{1-\overline{\rho}'\overline{\rho}},\overline{\rho}_1,\overline{\rho}_2,\overline{\rho}_3]'$. By definition, the market price of risk Θ_ν satisfies

$$\Sigma^{1/2}\Theta_\nu L = \mu^e\,, \tag{IA.C8}$$

from which

$$\Theta_\nu = \Sigma^{-1/2\prime}\mu^e L' + Y^{1/2}\nu\,, \tag{IA.C9}$$

where $\Sigma^{-1/2}=SY^{1/2}$ and ν is a 3×4 matrix-valued process such that $\nu L = 0_{3\times 1}$, that is $\nu = [-\overline{\nu}\frac{\overline{\rho}}{\sqrt{1-\overline{\rho}'\overline{\rho}}},\overline{\nu}]$. $\overline{\nu}$ is a 3×3 matrix that prices the shocks that drive the Wishart state variable Y.

It turns out, that the value function can be written in the form:

$$J(x,Y_0) = x^\gamma \inf_\nu \frac{1}{1-\gamma}\mathbb{E}\left[\xi_\nu(T)^{\frac{\gamma-1}{\gamma}}\right]^\gamma - \frac{1}{1-\gamma} = \frac{x^{1-\gamma}\widehat{J}(0,Y_0)^\gamma - 1}{1-\gamma},$$

where

$$\begin{aligned}\mathbb{E}\left[\xi_\nu(T)^{\frac{\gamma-1}{\gamma}}\right] &= \mathbb{E}^\gamma\left[e^{-\frac{\gamma-1}{\gamma}\int_0^T r(s)ds + \frac{1-\gamma}{2\gamma^2}tr\left(\int_0^T \Sigma(s)^{-1}ds\mu^e\mu^{e\prime} + \int_0^T Y(s)ds\overline{\nu}'\overline{\nu}(I_3+\frac{\overline{\rho}\overline{\rho}'}{1-\overline{\rho}'\overline{\rho}})\right)}\right]\\ &= \mathbb{E}^\gamma\left[e^{-\frac{\gamma-1}{\gamma}(r_0 + tr(\int_0^T Y(s)dsD)) + \frac{1-\gamma}{2\gamma^2}tr\left(\int_0^T Y(s)ds(S'\mu^e\mu^{e\prime}S + \overline{\nu}'\overline{\nu}(I_3+\frac{\overline{\rho}\overline{\rho}'}{1-\overline{\rho}'\overline{\rho}})\right)}\right],\end{aligned} \tag{IA.C10}$$

for a probability measure P^γ defined by the density

$$\frac{dP^\gamma}{dP} = e^{-tr\left(\frac{\gamma-1}{\gamma}\int_0^T \Theta_\nu'(s)dB + \frac{1}{2}\frac{(\gamma-1)^2}{\gamma^2}\int_0^T \Theta_\nu'(s)\Theta_\nu(s)ds\right)}.$$

The dynamics of Y under the probability P^γ are

$$\begin{aligned}dY(t) = &\left[\Omega\Omega' + \left(M - \frac{\gamma-1}{\gamma}Q'(\overline{\rho}\mu^{e\prime}S+\overline{\nu}')\right)Y(t) + Y(t)\left(M - \frac{\gamma-1}{\gamma}Q'(\overline{\rho}\mu^{e\prime}S+\overline{\nu}')\right)'\right]dt\\ &+ Y^{1/2}(t)dB^\gamma(t)Q + Q'dB(t)^{\gamma\prime}Y^{1/2}(t).\end{aligned} \tag{IA.C11}$$

These dynamics are affine in Y. It follows that the function $\widehat{J}$ is a solution of the following HJB equation:

$$0 = \frac{\partial\widehat{J}}{\partial t} + \inf_{\overline{\nu}}\left\{\mathcal{A}\widehat{J} + \widehat{J}\left[-\frac{\gamma-1}{\gamma}(r_0 + tr(YD)) + \frac{1-\gamma}{2\gamma^2}tr\left(Y\left(S'\mu^e\mu^{e\prime}S + \overline{\nu}'\overline{\nu}\left(I_3 + \frac{\overline{\rho}\overline{\rho}'}{1-\overline{\rho}'\overline{\rho}}\right)\right)\right)\right]\right\}, \tag{IA.C12}$$

subject to the terminal condition $\widehat{J}(T,Y)=1$, where $\mathcal{A}$ is the infinitesimal generator of the matrix-valued diffusion (IA.C11), which is given by

$$\begin{aligned}\mathcal{A} = &\,tr\left(\left(\Omega\Omega' + \left(M - \frac{\gamma-1}{\gamma}Q'(\overline{\rho}\mu^{e\prime}S+\overline{\nu}')\right)Y + Y\left(M - \frac{\gamma-1}{\gamma}Q'(\overline{\rho}\mu^{e\prime}S+\overline{\nu}')\right)'\right)\mathcal{D}\right)\\ &+ tr(2Y\mathcal{D}Q'Q\mathcal{D}).\end{aligned} \tag{IA.C13}$$

The generator is affine in Y. As in the proof of Proposition 2, the optimality condition for the optimal control $\overline{\nu}$ yields

$$\overline{\nu} = -\gamma \left(\frac{\mathcal{D}\widehat{J}}{\widehat{J}} + \frac{\mathcal{D}\widehat{J}'}{\widehat{J}} \right) Q' \left(I_3 + \frac{\overline{\rho}\overline{\rho}'}{1 - \overline{\rho}'\overline{\rho}} \right)^{-1}. \tag{IA.C14}$$

Note that $\left(I_3 + \frac{\overline{\rho}\overline{\rho}'}{1 - \overline{\rho}'\overline{\rho}} \right)^{-1} = I_3 - \overline{\rho}\overline{\rho}'$. Substituting the expression for $\overline{\nu}$ into equation (IA.C12), we obtain the following partial differential equation for $\widehat{J}$:

$$\begin{aligned}
-\frac{\partial \widehat{J}}{\partial t} = tr &\left(\left(\Omega\Omega' + \left(M - \frac{\gamma - 1}{\gamma} Q' \overline{\rho} \mu^{e\prime} S \right) Y + Y \left(M - \frac{\gamma - 1}{\gamma} Q' \overline{\rho} \mu^{e\prime} S \right)' \right) \mathcal{D} + 2Y\mathcal{D}Q'Q\mathcal{D} \right) \widehat{J} \\
&+ \frac{\gamma - 1}{\gamma} \widehat{J} \left(-r_0 - tr(YD) - \frac{tr(YS'\mu^e \mu^{e\prime} S)}{2\gamma} \right) \\
&- \frac{1 - \gamma}{2} \widehat{J}\, tr \left((I_3 - \overline{\rho}\overline{\rho}') Y \left(\frac{\mathcal{D}\widehat{J}}{\widehat{J}} + \frac{\mathcal{D}\widehat{J}'}{\widehat{J}} \right) Q'Q \left(\frac{\mathcal{D}\widehat{J}}{\widehat{J}} + \frac{\mathcal{D}\widehat{J}'}{\widehat{J}} \right)' \right),
\end{aligned}$$

subject to the boundary condition $\widehat{J}(\Sigma, T) = 1$. The affine structure of this problem suggests an exponentially affine functional form for its solution:

$$\widehat{J}(t, \Sigma) = \exp(B(t, T) + tr(A(t, T)Y),$$

for some state-independent coefficients $B(t, T)$ and $A(t, T)$. After inserting this functional form into the differential equation for $\widehat{J}$, the guess can be easily verified. The coefficients B and A are the solutions of the following system of Riccati equations:

$$\begin{aligned}
-\frac{dB}{dt} &= tr(A\Omega\Omega') - \frac{\gamma}{\gamma - 1} r_0, \\
-tr \left(\frac{dA}{dt} Y \right) &= tr \left(\Gamma' AY + A\Gamma Y + 2AQ'QAY - \frac{1 - \gamma}{2} (A' + A) Q' (I_3 - \overline{\rho}\overline{\rho}') Q (A' + A) Y + CY \right),
\end{aligned}$$

with terminal conditions $B(T, T) = 0$ and $A(T, T) = 0_{3\times 3}$, where

$$\Gamma = M - \frac{\gamma - 1}{\gamma} Q' \overline{\rho} \mu^{e\prime} S \tag{IA.C15}$$

$$C = \frac{1 - \gamma}{2\gamma^2} S' \mu^e \mu^{e\prime} S - \frac{1 - \gamma}{\gamma} D. \tag{IA.C16}$$

Explicit solutions for $B(t, T)$ and $A(t, T)$ are computed as in the proof of Proposition 2. By the same argument applied in the Proof of Proposition 3, the following equality must hold:

$$X^*(t)\, tr \left(\begin{bmatrix} \pi_1 & \pi_2 \end{bmatrix} \Sigma^{1/2} d\mathcal{B} L \right) = X^*(t)\, tr \left(\frac{1}{\gamma} \Theta'_{\nu^*} d\mathcal{B} + \frac{\mathcal{D}\widehat{J}'}{\widehat{J}} \left(Y^{1/2} d\mathcal{B} UQ + Q'U'd\mathcal{B}'Y \right) \right). \tag{IA.C17}$$

where matrix U is given by

$$U = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

This implies

$$L \begin{bmatrix} \pi_1 & \pi_2 \end{bmatrix} \Sigma^{1/2} = \frac{1}{\gamma} \left(L\mu^{e\prime} \Sigma^{-1/2} + \nu' Y^{1/2} \right) + 2UQAY^{1/2}.$$

Pre-multiplying both sides by L', post-multiplying them by $\Sigma^{-1/2\prime}$, and recalling that $L'\nu' = 0_{1\times 3}$ and $\Sigma^{-1/2} = SY^{1/2}$, we conclude that portfolio weight $\pi = (\pi_1, \pi_2)'$ is

$$\pi = \frac{1}{\gamma} \Sigma^{-1} \mu^e + 2\Sigma^{-1} SAQ' \overline{\rho}.$$

This concludes the proof of Proposition IA.C1:. □

The optimal policy (IA.C7) consists of a myopic and an intertemporal hedging portfolio, which are both proportional to the stochastic inverse covariance matrix. As noted by Chacko and Viceira (2005), in the univariate setting the relative size of the hedging and myopic demands is independent of the current level of volatility. This property also holds in the multivariate case, in the sense that both policies are proportional to the inverse covariance matrix Σ^{-1}.

We investigate the empirical implications of this specification in a scenario where, for simplicity, a constant interest rate ($D = 0$) has been assumed. This setting is the exact multivariate extension of the univariate model considered in Chacko and Viceira (2005). We use the same basic GMM estimation procedure and the same data used for the empirical application in the main text,[5] but we now apply it to the information matrix Σ^{-1}. The GMM moment restrictions for the variance-covariance matrix process are replaced by those for the precision process, which is assumed to follow a Wishart diffusion process. Table CI, Panel A, presents estimation results for the model with a constant risk premium. Panel B summarizes the estimated hedging demands.

Insert Table IA.CI about here

The myopic portfolio is time varying, via the variation of the inverse covariance matrix Σ^{-1}. This time variation is also partly reflected in the time variation of hedging demands. All in all, the absolute size of total hedging demands is comparable to that obtained in the main text for a constant market price of variance-covariance risk. For example, for a risk aversion parameter $\gamma = 6$ and an investment horizon of $T = 5$ years, the average hedging demand is approximately 23% of the myopic portfolio. Similar demands obtain for higher risk aversions and investment horizons.

D. Discrete-time Solution and Portfolio Constraints

D.1 Discrete-time Solution

In our model, the optimal dynamic trading strategy is given by a portfolio that must be rebalanced continuously over time. In practice, this can at best be an approximation, because trading is only possible at discrete trading dates. Moreover, transaction costs, liquidity constraints, or policy disclosure considerations might further constrain investors from frequent portfolio rebalancing. Even if we do not model these frictions explicitly in our setting, it is interesting to study the impact of discrete trading on the optimal hedging strategy in the context of our model.

Several studies have found that, as long as the investment opportunity set does not contain derivatives, the gains/losses of the optimal discrete-time portfolio policy with respect to a naively discretized continuous-time policy are small. See, for instance, Campbell et al. (2004) and Branger, Breuer, and Schlag (2006). We study whether similar conclusions hold in our multivariate portfolio choice setting. We consider the exact discrete-time process implied by the continuous-time model (1) to (4) of the main text, in which observations are generated at fixed, evenly spaced, points in time. The parameters of the continuous-time model have been estimated by GMM using the exact discrete-time moments of this process. The moments are easily obtainable in closed form for each sampling frequency because the Wishart process allows for aggregation over time. By construction, the estimated parameters are then consistent with the discrete time transition density of the process, which is the one relevant to study optimal portfolio choice in discrete-time.

The discrete-time portfolio choice problem does not allow for closed-form solutions. Therefore, we rely on standard numerical methods to compute the optimal portfolio strategies. Table IA.DI presents the total hedging demands in S&P500 Futures (π_1) and Treasury Futures (π_2), as fractions of the myopic demand. The transition density used for the discrete time portfolio optimization is the one implied by the estimated continuous time model with monthly returns, realized volatilities, and realized correlations.

Insert Table IA.DI about here

We focus on optimal portfolios that can be rebalanced monthly, but we also compute optimal strategies using a weekly and daily rebalancing frequency in order to verify the convergence of our numerical solution to the continuous-time portfolio problem solution. At a daily frequency, the hedging demands in the discrete-time model are virtually indistinguishable from the continuous-time hedging demands reported in Table III of the main text. Consistent with the findings in the literature, the discrete-time optimal hedging demands for the monthly frequency are close to the hedging demands computed from the continuous-time model: the mean absolute difference between the hedging demands using daily and monthly rebalancing is less than 10% of the hedging demand implied by a monthly rebalancing frequency. These findings suggest that the main implications derived from the continuous-time multivariate portfolio choice solutions are realistic even in the context of monthly rebalancing.

D.2 Portfolio Constraints

Portfolio constraints are useful to avoid unrealistic portfolio weights, which can potentially arise due to some extreme assumptions on expected returns, volatilities, and correlations, or from inaccurate point estimates of the model parameters. The empirical results of the previous sections can imply, for instance, levered portfolios in settings of low risk aversion. For instance, for a relative risk aversion of $\gamma = 2$, the optimal portfolio of an investor with horizon $T = 5$ years implies an investment of approximately 260% of the total wealth in stocks and 170% in bonds. Intuitively, constraints on short selling or on the portfolio VaR tend to constrain the investor from selecting optimal portfolios that are excessively levered. Therefore, it is interesting to study these types of portfolio constraints and their impact on the volatility and correlation hedging demands in our setting. We solve the discrete-time portfolio choice problem in the last section and additionally impose, in two separate steps, short- selling and VaR constraints. In order to quantify the correlation and volatility hedging components, we numerically compute the projection of the total hedging demand on the implied elasticity of the indirect marginal utility of wealth with respect to volatilities and covariances.

In the first exercise, we consider state-independent constraints on the optimal portfolio weights. For every fraction π_i of total wealth invested in the risky asset i, we first enforce a short-selling constraint $\pi_i \geq 0$. In a second step, we also consider a less severe position limit $\pi_i \geq -1$. Table IA.DII presents the optimal volatility and covariance hedging demands implied by these two settings. Note that even in cases where the current constraint might not be binding, the optimal hedging strategy is different from the one implied by the unconstrained solution. This feature exists because the future opportunity set is restricted by the fact that the constraint might be binding, with some probability, in the future. The indirect marginal utility of wealth in the constrained problem depends on the strength of this effect. Therefore, the optimal intertemporal hedging demand is different.

Insert Table IA.DII about here

Table IA.DII shows that the more severe the constraint is, the smaller are the absolute demands for volatility and covariance hedging as a percent of the myopic portfolio. However, the impact of the constraint is quite moderate, even in the short-selling case, and does not greatly influence the relative size of the hedging demands against volatility and covariance risk across assets. For instance, for an investment horizon of $T = 10$ years and a risk aversion of $\gamma = 2$, the average covariance (volatility) hedging demand is 10.5% (7%) in the unconstrained case and 8.5% (6.5%) in the setting with short selling constraints. For a higher risk aversion of $\gamma = 8$, the average covariance (volatility) hedging demand is 13.25% (10.25%) in the unconstrained case and 10.75% (9%) in the setting with short-selling constraints. These findings are consistent with the state-independent nature of the constraint used, which is not a function of the conditional covariance matrix of returns. The slightly larger percentage decrease in the hedging demands of low risk-aversion investors in the constrained case is mainly due to their large myopic demands in the unconstrained portfolio problem.

The results are different when we study the effects of (state-dependent) VaR constraints. At each trading date, we impose a constant upper bound on the VaR of the optimally invested wealth at the next trading date. We use a VaR at a confidence level of 99%. Since the VaR is computed for a monthly rebalancing frequency and investment horizons longer than one month, the VaR constraint is dynamically updated, as in Cuoco, He, and Isaenko (2008). Table IA.DIII summarizes our findings for the optimal VaR-constrained portfolios. For computational tractability of our numerical solutions, we focus on investment horizons up to $T = 2$ years.

Insert Table IA.DIII about here

The VaR constraint has a more significant effect on the optimal portfolios of investors with low risk aversion, which are those with the largest exposure to risky assets in the unconstrained setting. For instance, for a risk aversion coefficient of $\gamma = 2$ and an investment horizon of $T = 2$ years, the mean total allocation to stocks (bonds) shrinks from approximately 250% (160%) to about 175% (115%) of the total wealth. At the same time, the relative importance of the covariance hedging demand increases: even for a moderate investment horizon of $T = 2$ years and a low risk aversion of $\gamma = 2$, the correlation and volatility hedging demands are on average 11% and 7% of the myopic portfolio, respectively. With the same choice of parameters, the corresponding hedging demands in the unconstrained case are 7.7% and 10.7%, respectively. For a higher risk aversion of $\gamma = 8$ and the same investment horizon, the covariance hedging demand is on average about 11% of the myopic portfolio both in the VaR-constrained and VaR-unconstrained cases.

The VaR-constrained investor dislikes more volatile or extreme portfolio values than the unconstrained agent does, since (ceteris paribus) the VaR constraint becomes more restrictive when the volatility on the optimally invested portfolio increases. It follows that the investor is more concerned about the total volatility of the portfolio, which can cause the VaR constraint to be hit with a probability that is too large. Therefore, the VaR-constrained investor reduces the size of the myopic demand. Furthermore, since changes in correlation have a first-order impact on the VaR of the portfolio, the investor increases the covariance hedging demand, exploiting the spanning properties of the risky assets. Thus, in this setting, which is relevant for institutions subject to capital requirement or for asset managers with self-imposed risk management constraints, the impact of covariance risk is economically significant.

E. Additional Empirical Results

Figure IA.E1. The Effect of the Investment Horizon

Figure IA.E1 reports intertemporal hedging demands for the S&P500 Index futures and the 30-year Treasury futures, as functions of the investment horizon, using the GMM parameter estimates for the underlying opportunity set dynamics reported in Table I of the main text.

Figure IA.E2. The Effect of the Risk Aversion Parameter

Figure IA.E2 plots hedging demands as a function of the coefficient of Relative Risk Aversion.

Table IA.EI. Estimation Results for Univariate Stochastic Volatility Models

We compare the portfolio implications of our setting with those of univariate portfolio choice models with stochastic volatility; see Heston (1993) and Liu (2001), among others. These models are nested in our setting in the special case in which the dimension of the investment opportunity set is set equal to one. For each risky asset in our data set, we estimate these univariate stochastic volatility models by GMM. The moment restrictions employed are the univariate counterpart of the moment conditions used in the estimation of the multivariate model. Panel A of Table IA.EI presents parameter estimates, whereas Panel B reports the estimated volatility hedging demands as a percentage of the myopic portfolio.

Table IA.EII. Estimation Results for Model with Three Risky Assets

Using GMM we estimate the three-dimensional version of model (1) to (4) in the main text obtained by including also the Nikkei225 Index futures contract in the opportunity set consisting of the S&P500 futures and the 30-year Treasury futures contracts. We use monthly time series of returns, realized volatilities, and realized covariances for these three risky assets. GMM moment restrictions are obtained in closed form as for the bivariate case above using the properties of the Wishart process. It is also straightforward to extend the proofs of Propositions 2 and 3 in the main text to cover the general setting with n risky assets. With these results, we compute the estimated optimal portfolios for the model with three risky assets. Table IA.EII, Panel A, presents the results of our GMM model estimation. The implied hedging demands for covariance and pure volatility hedging on each asset are given in Panel B.

Table IA.EIII. Estimation Results for Model with Two Risky Assets Using Daily Data

Table IA.EIII reports estimates for the parameters of model (1) to (4) in the main text, obtained with daily data and using the GMM procedure discussed in Internet Appendix B.

Table IA.EIV. Optimal Hedging Demands with Two Risky Assets Using Weekly Data

Table IA.EIV reports estimated optimal covariance and volatility hedging demands obtained using the weekly parameter estimates reported in Table I of the main text.

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Notes

[1]Remember that $W = \mathcal{B}L$.

[2]Strictly speaking, this holds for $\gamma \in (0,1)$. For $\gamma > 1$, minimizations are replaced by maximizations and all formulas follow with the same type of arguments.

[3]In this way, local asymmetries in the covariance matrix dynamics can be introduced in the model. To model asymmetric correlations across regimes, Ang and Bekaert (2002) use an i.i.d. regime-switching setting, in which one of the regimes is characterized by greater correlations and volatilities.

[4]A possible choice for S is a 2×3 selection matrix, for example,

$$S = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

In this case, $SS' = id_{2\times 2}$ and SYS' is the 2×2 upper diagonal sub-block of Y.

[5]Discussed in Internet Appendix B.

Table IA.CI

Estimation Results and Hedging Demands for the Model with Constant Risk Premia

Panel A: We report parameter estimates, Hansen's statistics, and hedging demands for the following model specification:

$$
\begin{aligned}
dS(t) &= I_S\, \mu\, dt + I_S\, Y^{-1/2}(t)(dB(t)\overline{\rho} + \sqrt{1-\overline{\rho}'\overline{\rho}}\, dZ(t)) \\
dY(t) &= \left[\Omega\Omega' + MY(t) + Y(t)M'\right] dt + Y^{1/2}(t)dB(t)Q + Q'dB(t)'Y^{1/2}(t),
\end{aligned}
$$

$S(t)$ is the two-dimensional vector of the prices of S&P500 Index and 30-year Treasury bond futures. μ is a bivariate vector of constants and the interest rate r is also constant. $Y(t)$ models the information matrix $\Sigma(t)^{-1}$ and follows a Wishart diffusion. $B(t)$ is a 2×2 matrix of standard Brownian motions and $Z(t)$ is a 2×1 vector of Brownian motions independent of $B(t)$. Vector $\overline{\rho}$ and matrices M and Q are the remaining model parameters. Parameters are estimated with the same GMM method discussed in Appendix B, that is now applied to the information matrix $Y = \Sigma^{-1}$ sampled at a monthly frequency. An asterisk denotes parameter estimates that are not significant at the 5% significance level. Panel B: Optimal hedging demands in percentages of the myopic portfolio are given for different investment horizons and relative risk aversion parameters. Each entry of the array of Panel B is a two-dimensional vector, the first component of which is the hedging demand for the S&P500 Index futures, while the second one is the hedging demand for the 30-year Treasury futures.

Panel A

	M		Q		$\overline{\rho}$	μ
point estimates (standard errors)	−0.149	0.114*	0.706	0.494*	0.381	0.0616
	(0.074)	(0.081)	(0.34)	(0.312)	(0.161)	(0.008)
	0.070	−0.112	0.806	0.641*	0.392	0.0114
	(0.036)	(0.055)	(0.371)	(0.591)	(0.189)	(0.0009)
p-value for Hansen's J-test	0.254					

Panel B

RRA T	$3m$	$6m$	$1y$	$2y$	$5y$	$7y$	$10y$
2	−0.034	−0.061	−0.111	−0.151	−0.172	−0.173	−0.174
	−0.036	−0.053	−0.095	−0.135	−0.144	−0.145	−0.146
6	−0.049	−0.105	−0.175	−0.240	−0.251	−0.252	−0.253
	−0.048	−0.090	−0.160	−0.207	−0.213	−0.214	−0.214
8	−0.057	−0.115	−0.182	−0.258	−0.262	−0.263	−0.264
	−0.053	−0.104	−0.164	−0.212	−0.224	−0.224	−0.224
11	−0.061	−0.119	−0.189	−0.260	−0.263	−0.264	−0.265
	−0.055	−0.108	−0.170	−0.219	−0.228	−0.229	−0.230
16	−0.063	−0.120	−0.190	−0.261	−0.270	−0.271	−0.271
	−0.063	−0.110	−0.171	−0.220	−0.230	−0.231	−0.231
21	−0.065	−0.121	−0.191	−0.262	−0.272	−0.273	−0.273
	−0.064	−0.111	−0.172	−0.221	−0.231	−0.232	−0.232
41	−0.065	−0.121	−0.192	−0.262	−0.273	−0.274	−0.275
	−0.065	−0.112	−0.173	−0.221	−0.233	−0.235	−0.236

Table IA.DI

Optimal Hedging Demands in the Discrete-Time Model

Using standard numerical dynamic programming methods, we compute optimal hedging demands in percentages of the myopic portfolio for the exact discretization of the continuous-time model (1) to (4) of the main text, for different investment horizons and relative risk aversion parameters. The parameters used to compute the exact discrete-time transition density of the model are the monthly estimates in Table II of the main text. We compute optimal discrete-time hedging demands for a daily (d), a weekly (w), and a monthly (m) rebalancing frequency, and denote by π_1 and π_2 the hedging demands for the S&P500 Index and the 30-year Treasury bond futures, respectively.

RRA	*T*	*3m*			*6m*			*1y*			*2y*		
			π_1	π_2		π_1	π_2		π_1	π_2		π_1	π_2
2		d	0.0304	0.0344	d	0.0541	0.0650	d	0.0910	0.1151	d	0.1250	0.1661
		w	0.0295	0.0401	w	0.0561	0.0702	w	0.0917	0.1189	w	0.1243	0.1715
		m	0.0291	0.0449	m	0.0570	0.0759	m	0.0918	0.1240	m	0.1248	0.1879
			π_1	π_2		π_1	π_2		π_1	π_2		π_1	π_2
8		d	0.0525	0.0589	d	0.0972	0.1123	d	0.1550	0.1915	d	0.1975	0.2543
		w	0.0515	0.0605	w	0.1021	0.1162	w	0.1545	0.1955	w	0.1969	0.2636
		m	0.0525	0.0632	m	0.1078	0.1208	m	0.1533	0.1803	m	0.1966	0.2566
			π_1	π_2		π_1	π_2		π_1	π_2		π_1	π_2
21		d	0.0573	0.0640	d	0.1069	0.1209	d	0.1686	0.2045	d	0.2111	0.2685
		w	0.0569	0.0641	w	0.1134	0.1259	w	0.1705	0.2078	w	0.2099	0.2645
		m	0.0580	0.0665	m	0.1266	0.1365	m	0.1761	0.1993	m	0.2076	0.2705

Table IA.DII

Optimal Hedging Demands in the Discrete-time Model with Short-selling Constraints

Using standard numerical dynamic programming methods, we compute optimal hedging demands as a percentage of the myopic portfolio for the exact discretization of the continuous-time model (1) to (4) of the main text, when short-selling constraints are applied, for different investment horizons and relative risk aversion parameters. The parameters used to compute the exact discrete-time transition density of the model are the monthly estimates in Table II of the main text, and the rebalancing frequency is monthly. We denote by π_1 and π_2 the hedging demands for the S&P500 Index and the 30-year Treasury bond futures, respectively, and distinguish the cases u, c_1, and c_2 corresponding to the unconstrained solution, the solution for a position limit of the form $\pi \geq -1$, and the solution in the short-selling constrained case ($\pi \geq 0$), respectively. Total hedging demands are decomposed into covariance and volatility hedging components by means of a cross-sectional regression of simulated hedging demands on the wealth-scaled ratios of simulated indirect marginal utilities of covariance and variances.

Covariance hedging

RRA	*T*	3m		6m		1y		2y		5y		7y		10y	
		π_1	π_2	π_1	π_2	π_1	π_2	π_1	π_2	π_1	π_2	π_1	π_2	π_1	π_2
2	u	0.0071	0.0352	0.0168	0.0521	0.0319	0.0814	0.0406	0.1281	0.0512	0.1293	0.0571	0.1382	0.0562	0.1394
	c_1	0.0081	0.0254	0.0154	0.0448	0.0298	0.0695	0.0393	0.1280	0.0515	0.1091	0.0528	0.1125	0.0498	0.1121
	c_2	0.0054	0.0253	0.0136	0.0387	0.0283	0.0591	0.0385	0.0932	0.0486	0.1043	0.0513	0.1084	0.0514	0.1089
		π_1	π_2	π_1	π_2	π_1	π_2	π_1	π_2	π_1	π_2	π_1	π_2	π_1	π_2
8	u	0.0140	0.0435	0.0298	0.0924	0.0452	0.1244	0.0619	0.1859	0.0603	0.1924	0.0612	0.1929	0.0612	0.1929
	c_1	0.0100	0.0322	0.0291	0.0812	0.0425	0.1021	0.0592	0.1635	0.0532	0.1765	0.0581	0.1818	0.0581	0.1819
	c_2	0.0084	0.0316	0.0215	0.0760	0.0392	0.0951	0.0496	0.1486	0.0524	0.1522	0.0525	0.1533	0.0525	0.1534
		π_1	π_2	π_1	π_2	π_1	π_2	π_1	π_2	π_1	π_2	π_1	π_2	π_1	π_2
21	u	0.0165	0.0484	0.0371	0.0842	0.0510	0.1328	0.0671	0.1828	0.0651	0.1985	0.0662	0.2001	0.0679	0.2003
	c_1	0.0139	0.0491	0.0363	0.0715	0.0481	0.1296	0.0623	0.1768	0.0632	0.1812	0.0679	0.1858	0.0679	0.1858
	c_2	0.0125	0.0367	0.0342	0.0683	0.0413	0.1198	0.0556	0.1541	0.0561	0.1599	0.0588	0.1615	0.0588	0.1615

Volatility hedging

RRA	*T*	3m		6m		1y		2y		5y		7y		10y	
		π_1	π_2	π_1	π_2	π_1	π_2	π_1	π_2	π_1	π_2	π_1	π_2	π_1	π_2
2	u	0.0198	0.0076	0.0381	0.0190	0.0578	0.0330	0.0790	0.0459	0.0801	0.0588	0.0802	0.0589	0.0803	0.0591
	c_1	0.0171	0.0074	0.0342	0.0185	0.0538	0.0325	0.0750	0.0455	0.0765	0.0581	0.0766	0.0583	0.0766	0.0583
	c_2	0.0151	0.0067	0.0284	0.0183	0.0478	0.0318	0.0690	0.0420	0.0711	0.0571	0.0712	0.0572	0.0714	0.0574
		π_1	π_2	π_1	π_2	π_1	π_2	π_1	π_2	π_1	π_2	π_1	π_2	π_1	π_2
8	u	0.0354	0.0131	0.0755	0.0254	0.1031	0.0499	0.1287	0.0655	0.1344	0.0698	0.1345	0.0700	0.1345	0.0700
	c_1	0.0321	0.0118	0.0712	0.0252	0.0945	0.0489	0.1189	0.0631	0.1255	0.0677	0.1255	0.0677	0.1255	0.0677
	c_2	0.0290	0.0111	0.0673	0.0250	0.0891	0.0451	0.1121	0.0612	0.1150	0.0633	0.1150	0.0633	0.1150	0.0633
		π_1	π_2	π_1	π_2	π_1	π_2	π_1	π_2	π_1	π_2	π_1	π_2	π_1	π_2
21	u	0.0389	0.0154	0.0711	0.0294	0.1201	0.0525	0.1375	0.0705	0.1439	0.0710	0.1439	0.0710	0.1439	0.0710
	c_1	0.0366	0.0149	0.0657	0.0291	0.1088	0.0523	0.1301	0.0701	0.1354	0.0704	0.1354	0.0704	0.1354	0.0704
	c_2	0.0312	0.0143	0.0621	0.0281	0.1061	0.0511	0.1145	0.0671	0.1201	0.0691	0.1201	0.0691	0.1201	0.0691

Table IA.DIII

Optimal Hedging Demands in the Discrete-time Model with VaR constraints

This table reports optimal VaR-constrained volatility and covariance hedging demands in percentages of the myopic portfolio for the exact discretization of the continuous-time model (1) to (4) of the main text, as a function of different investment horizons and relative risk aversion parameters. The parameters used to compute the exact discrete-time transition density of the model are the monthly estimates in Table II of the main text, and the rebalancing frequency is monthly. As in Cuoco, He, and Isaenko (2008), the VaR constraint is updated at each trading date, by imposing a constant upper bound on the 99%-VaR of next-trading-date wealth. Total hedging demands are decomposed into covariance and volatility hedging components by means of a cross-sectional regression of simulated hedging demands on the wealth-scaled ratios of simulated indirect marginal utilities of variances and covariances. Each entry of the two arrays in the table is a two-dimensional vector, the first component of which is the hedging demand for the S&P500 Index futures, while the second one is the hedging demand for the 30-year Treasury bond futures.

Volatility Hedging

RRA	T	$3m$	$6m$	$1y$	$2y$
2		0.023	0.048	0.064	0.086
		0.014	0.026	0.040	0.058
8		0.028	0.052	0.069	0.092
		0.019	0.031	0.045	0.061
21		0.031	0.055	0.072	0.095
		0.023	0.032	0.047	0.062

Covariance Hedging

RRA	T	$3m$	$6m$	$1y$	$2y$
2		0.019	0.035	0.048	0.069
		0.043	0.073	0.101	0.130
8		0.021	0.041	0.051	0.082
		0.046	0.081	0.102	0.138
21		0.032	0.052	0.072	0.082
		0.059	0.097	0.121	0.149

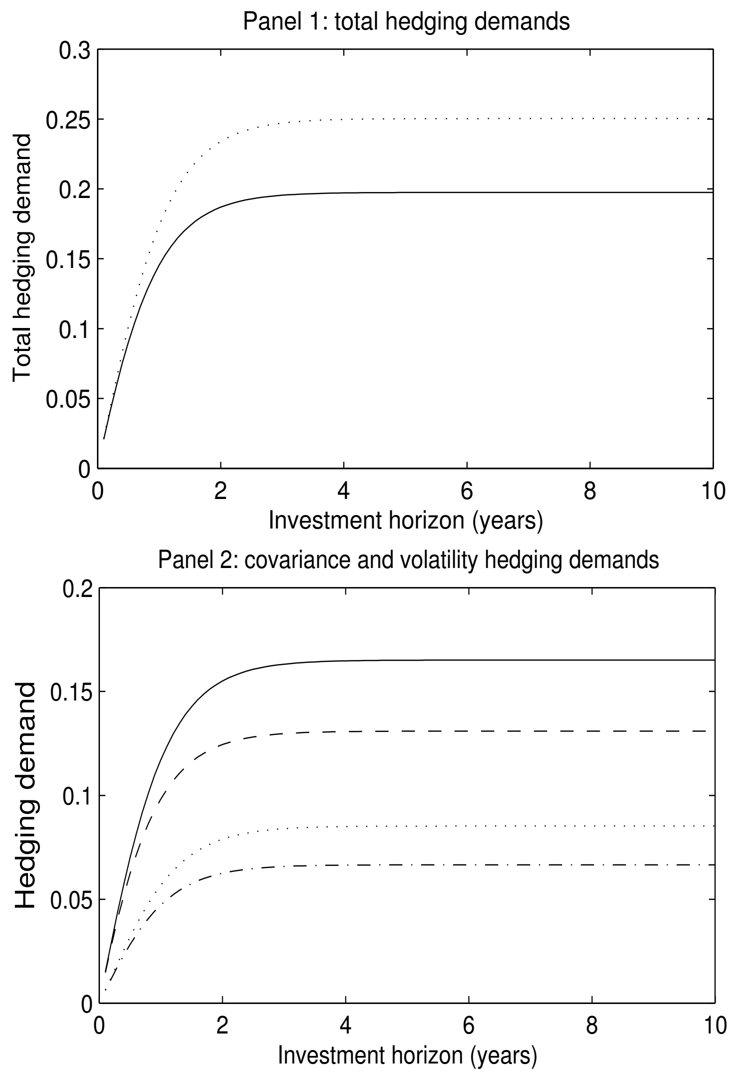

Figure IA.E1. The effect of the investment horizon. Panel 1: Total hedging demands for the S&P500 Index futures (solid line) and 30-year Treasury futures (dotted line) as a percentage of the Merton myopic portfolio are plotted as a function of the investment horizon (in years). These hedging demands are computed using the monthly parameter estimates in Table I of the main text, for a relative risk aversion parameter of $\gamma = 6$. Panel 2: Volatility hedging and covariance hedging demands for the 30-year Treasury bond futures (dotted and solid lines, respectively) and the S&P500 Index futures (dashed and dashed-dotted lines, respectively) are plotted as functions of the investment horizon (in years). Both hedging demands are expressed as a percentage of the Merton myopic portfolio. The same parameters as for Panel 1 are used to computed them.

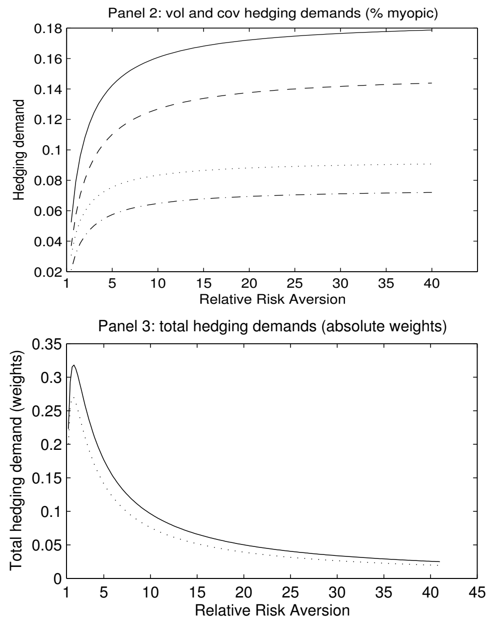

Figure IA.E2. The effect of the risk aversion parameter. Panel 1: Total hedging demands for the S&P500 Index futures (solid line) and 30-year Treasury futures (dotted line) as a percentage of the Merton myopic portfolio are plotted as functions of the relative risk aversion coefficient for a fixed investment horizon of five years. To compute these policies, we use the monthly parameters estimates in Table I of the main text. Panel 2: Volatility hedging and covariance hedging demands for the 30-year Treasury bond futures (dotted and solid lines, respectively) and the S&P500 Index futures (dashed and dashed-dotted lines, respectively) as a percentages of the Merton myopic portfolio are plotted as functions of the relative risk aversion coefficient. The same parameters as in Panel 1 are used to compute these policies. Panel 3: Same plots as in Panel 1, but with percentage hedging demands replaced by actual hedging portfolio weights. Panel 4: Total portfolio weights for covariance hedging (solid line) and for volatility hedging (dotted line), aggregated over risky assets, are plotted as functions of the Relative risk aversion parameter. The same parameters as in Panel 1 are used to compute these policies.

Table IA.EI

Estimation Results and Hedging Demands for Univariate Stochastic Volatility Models

Panel A: We report point estimates and standard errors (in parentheses) for the parameters of the following univariate stochastic volatility model:

$$dS_t = S_t(r + \lambda\sigma_t^2)dt + \sigma_t(\rho dW_t + \sqrt{1-\rho^2}dZ_t)$$
$$d\sigma_t^2 = (k\,b^2 + 2\,m\,\sigma_t^2)dt + 2\,b\,\sigma_t dW_t. \tag{1T}$$

S_t is the futures price of either the S&P500 futures, the 30-year Treasury bond futures, or the Nikkei 225 Index futures. σ_t is the stochastic volatility process of returns, modeled by a Heston (1993)-type model. W_t and Z_t are independent scalar Brownian motions and (k, λ, ρ, b, m) is the vector of parameters of interest. We estimate model (1T) by GMM using monthly time series of returns and realized volatilities for the S&P500 futures, 30-year Treasury bond futures, and Nikkei 225 Index futures returns. Panel B: We compute optimal (volatility) hedging demands for the univariate stochastic volatility model (1T), as a percentage of the myopic portfolio, using the parameter estimates in Panel A and for different investment horizons and relative risk aversion coefficients. The last column reports optimal myopic demands. The notation $S\&P500$, $Trea$, and $Nik225$ corresponds to the hedging demands in the univariate models for the S&P500 Index futures, the 30-year Treasury Bond futures, and the Nikkei225 Index futures, respectively.

Panel A

	k	m	b	ρ	λ
S&P500	1.18	−2.39	0.36	−0.88	0.72
	(0.36)	(0.42)	(0.08)	(0.05)	(0.21)
Treasury	2.45	−2.10	0.29	−0.56	1.05
	(0.84)	(0.24)	(0.07)	(0.04)	(0.34)
Nikkei	4.33	−2.82	−0.28	−0.67	0.64
	(1.16)	(0.55)	(0.08)	(0.19)	(0.19)

Panel B

RRA	T	$6m$		$1y$		$5y$		$10y$		Myopic demand	
		S&P500	0.022	S&P500	0.025	S&P500	0.027	S&P500	0.027	S&P500	0.360
2		Trea	0.018	Trea	0.021	Trea	0.023	Trea	0.024	Trea	0.525
		Nik225	0.012	Nik225	0.019	Nik225	0.023	Nik225	0.025	Nik225	0.320
		S&P500	0.034	S&P500	0.038	S&P500	0.041	S&P500	0.041	S&P500	0.180
4		Trea	0.028	Trea	0.032	Trea	0.034	Trea	0.034	Trea	0.262
		Nik225	0.016	Nik225	0.021	Nik225	0.031	Nik225	0.033	Nik225	0.160
		S&P500	0.041	S&P500	0.045	S&P500	0.048	S&P500	0.048	S&P500	0.090
8		Trea	0.033	Trea	0.038	Trea	0.040	Trea	0.040	Trea	0.131
		Nik225	0.019	Nik225	0.025	Nik225	0.036	Nik225	0.038	Nik225	0.080

Table IA.EII

Estimation Results and Hedging Demands for the Model with 3 Risky Assets

We present parameter estimates, Hansen's statistics and optimal hedging demands for model (1)-(4) with 3 risky assets. *Panel A:* We report parameter estimates for M, Q, λ and $\overline{\rho}$ (with standard errors in parentheses) in the returns dynamics (1)-(4), where $\Omega\Omega' = kQQ'$ for $k = 10$. The parameters are estimated using monthly returns, realized volatilities and correlations of S&P 500 index, 30-year US Treasury bond, and Nikkei 225 index future returns sampled at a monthly frequency. The GMM estimation procedure is similar to the one used to estimate the bivariate model and detailed moment restrictions are given in Appendix B. Parameters that are not significant at the 5% significance level are marked with an asterisk. *Panel B:* We report optimal covariance and volatility hedging demands in percentage of the myopic portfolio. Each entry of the array in Panel B consists of three components, the first of which is the demand for the S&P500 Index Futures, the second one the demand for the 30-year Treasury bond Futures and the third one the demand for the Nikkei 225 Index Futures, respectively.

Panel A

	M			Q			$\overline{\rho}$	λ
	−0.762 (0.293)	−0.251* (0.162)	0.390 (0.180)	0.005* (0.060)	0.064 (0.029)	0.069* (0.051)	−0.210 (0.090)	2.482 (0.380)
point estimates (standard errors)	0.511 (0.240)	−0.872 (0.281)	0.120* (0.110)	0.059* (0.048)	0.105* (0.074)	0.060* (0.100)	−0.230* (0.140)	2.327 (0.490)
	0.286* (0.153)	0.425 (0.212)	−0.968 (0.394)	0.070 (0.033)	0.055 (0.022)	0.004* (0.180)	−0.170 (0.075)	1.561 (0.160)
p-value for Hansen's J-test		0.115						

Panel B

Volatility Hedging

RRA	T	$3m$	$6m$	$1y$	$2y$	$5y$	$7y$	$10y$	$20y$	Myopic demand
2		0.005 0.007 0.004	0.009 0.011 0.008	0.017 0.016 0.014	0.027 0.020 0.023	0.034 0.021 0.029	0.035 0.021 0.030	0.035 0.021 0.030	0.035 0.021 0.030	1.2410 1.1635 0.7805
6		0.014 0.019 0.013	0.028 0.033 0.025	0.051 0.050 0.044	0.077 0.059 0.065	0.090 0.061 0.076	0.091 0.061 0.077	0.091 0.061 0.077	0.091 0.061 0.077	0.4137 0.3878 0.2602
8		0.018 0.026 0.018	0.037 0.045 0.034	0.066 0.065 0.059	0.097 0.076 0.083	0.110 0.078 0.093	0.110 0.078 0.094	0.110 0.078 0.094	0.110 0.078 0.094	0.3103 0.2909 0.1951
11		0.025 0.036 0.025	0.050 0.061 0.046	0.088 0.087 0.077	0.123 0.100 0.105	0.134 0.101 0.114	0.134 0.101 0.114	0.135 0.101 0.115	0.135 0.101 0.115	0.2256 0.2115 0.1419
16		0.037 0.052 0.035	0.071 0.087 0.065	0.122 0.121 0.106	0.158 0.133 0.136	0.167 0.134 0.143	0.167 0.134 0.144	0.167 0.134 0.144	0.167 0.134 0.144	0.1551 0.1454 0.0976
21		0.048 0.068 0.047	0.092 0.113 0.085	0.150 0.152 0.132	0.187 0.163 0.161	0.194 0.164 0.167	0.194 0.164 0.167	0.194 0.164 0.167	0.194 0.164 0.167	0.1182 0.1108 0.0743
41		0.092 0.130 0.090	0.166 0.200 0.153	0.242 0.250 0.215	0.275 0.260 0.237	0.278 0.260 0.240	0.279 0.260 0.240	0.279 0.260 0.240	0.279 0.260 0.240	0.0605 0.0568 0.0381

Covariance Hedging

RRA	T	$3m$	$6m$	$1y$	$2y$	$5y$	$7y$	$10y$	$20y$	Myopic demand
2		0.010 0.004 0.008	0.017 0.008 0.016	0.030 0.013 0.029	0.041 0.019 0.042	0.046 0.021 0.048	0.048 0.022 0.048	0.048 0.022 0.049	0.049 0.022 0.049	1.2410 1.1635 0.7805
6		0.029 0.013 0.026	0.050 0.025 0.046	0.087 0.041 0.079	0.117 0.058 0.114	0.125 0.061 0.133	0.130 0.065 0.138	0.130 0.065 0.138	0.130 0.065 0.138	0.4137 0.3878 0.2602
8		0.041 0.016 0.031	0.068 0.031 0.061	0.119 0.051 0.102	0.151 0.068 0.148	0.163 0.072 0.159	0.163 0.073 0.159	0.164 0.073 0.160	0.164 0.073 0.160	0.3103 0.2909 0.1951
11		0.054 0.022 0.044	0.102 0.040 0.085	0.158 0.071 0.144	0.182 0.087 0.184	0.195 0.094 0.191	0.195 0.095 0.191	0.196 0.096 0.192	0.196 0.096 0.192	0.2256 0.2115 0.1419
16		0.078 0.031 0.063	0.138 0.062 0.122	0.211 0.098 0.196	0.253 0.125 0.247	0.265 0.128 0.247	0.267 0.130 0.268	0.267 0.130 0.268	0.268 0.132 0.270	0.1551 0.1454 0.0976
21		0.095 0.041 0.082	0.178 0.083 0.159	0.269 0.128 0.251	0.291 0.145 0.310	0.310 0.152 0.321	0.310 0.154 0.322	0.310 0.154 0.322	0.310 0.154 0.323	0.1182 0.1108 0.0743
41		0.172 0.081 0.167	0.331 0.159 0.271	0.422 0.193 0.391	0.450 0.205 0.431	0.451 0.206 0.432	0.451 0.206 0.432	0.451 0.206 0.432	0.451 0.206 0.432	0.0605 0.0568 0.0381

Table IA.EIII

Estimation Results for the Model with Two Risky Assets Using Daily Data

This table shows estimated matrices M and Q and vectors λ and $\overline{\rho}$ for the returns dynamics (1) in the main text, under the Wishart variance covariance diffusion process:

$$d\Sigma(t) = (\Omega\Omega' + M\Sigma(t) + \Sigma(t)M')dt + \Sigma^{1/2}(t)dB(t)Q + Q'dB(t)'\Sigma^{1/2}(t),$$

where $\Omega\Omega' = kQ'Q$ and $k = 10$. Parameters are estimated by GMM using time series of returns and realized variance-covariance matrices for S&P 500 Index and 30-year Treasury bond futures returns, computed for a daily frequency. The detailed set of moment restrictions used for GMM estimation is given in Internet Appendix B. We report parameter estimates and their standard errors (in parentheses), together with the p-values for Hansen's J-test of overidentifying restrictions.

	M		Q		λ	$\overline{\rho}$
point estimates (*p-values*)	−1.098 (0.0002)	0.42 (0.001)	−0.16 (0.01)	0.028 (0.3435)	4.89 (0.03)	0.1296 (0.0035)
	0.21 (0.002)	−1.58 (0.0035)	0.0049 (0.4534)	0.103 (0.024)	5.54 (0.04)	−0.24 (0.0121)
p-value for Hansen's J-test	0.03					

Table IA.EIV

Optimal Hedging Demands in the Model with Two Risky Assets Using Weekly Data

This table shows optimal covariance and volatility hedging demands as a percentage of the myopic portfolio, for different investment horizons and relative risk aversion parameters. The last column of each panel reports the myopic portfolio. We compute these demands for the weekly parameters estimates reported in Table I of the main text. Each entry in the table is a vector with two components, nemely the demand for the S&P500 Index futures and the demand for the 30-year Treasury futures.

Covariance Hedging

RRA T	$3m$	$6m$	$1y$	$2y$	$5y$	$7y$	$10y$	Myopic demand
2	0.0186	0.0310	0.0441	0.0512	0.0523	0.0523	0.0523	2.3610
	0.0121	0.0202	0.0290	0.0334	0.0342	0.0342	0.0342	1.6585
6	0.0311	0.0520	0.0732	0.0835	0.0848	0.0848	0.0848	0.7870
	0.0203	0.0340	0.0480	0.0541	0.0554	0.0555	0.0555	0.5528
8	0.0327	0.0545	0.0768	0.0874	0.0888	0.0888	0.0888	0.5903
	0.0214	0.0357	0.0502	0.0571	0.0580	0.0580	0.0580	0.4146
11	0.0340	0.0568	0.0797	0.0906	0.0920	0.0920	0.0920	0.4293
	0.0222	0.0371	0.0521	0.0592	0.0610	0.0610	0.0610	0.3015
16	0.0351	0.0586	0.0822	0.0933	0.0947	0.0947	0.0947	0.2951
	0.0229	0.0383	0.0537	0.0618	0.0621	0.0621	0.0621	0.2073
21	0.0359	0.0595	0.0835	0.0947	0.0961	0.0961	0.0961	0.2249
	0.0233	0.0389	0.0545	0.0623	0.0626	0.0626	0.0626	0.1580
41	0.0366	0.0610	0.0855	0.0969	0.0983	0.0985	0.0985	0.1152
	0.0239	0.0400	0.0556	0.0633	0.0642	0.0642	0.0642	0.0809

Volatility Hedging

RRA T	$3m$	$6m$	$1y$	$2y$	$5y$	$7y$	$10y$	Myopic demand
2	0.0116	0.0188	0.0256	0.0289	0.0294	0.0294	0.0294	2.3610
	0.0194	0.0336	0.0500	0.0599	0.0617	0.0617	0.0617	1.6585
6	0.0195	0.0314	0.0425	0.0477	0.0477	0.0477	0.0477	0.7870
	0.0327	0.0564	0.0830	0.0977	0.0999	0.0999	0.0999	0.5528
8	0.0205	0.0330	0.0446	0.0494	0.0500	0.0500	0.0500	0.5903
	0.0343	0.0593	0.0871	0.1023	0.1046	0.1046	0.1046	0.4146
11	0.0213	0.0343	0.0463	0.0513	0.0518	0.0518	0.0518	0.4293
	0.0357	0.0617	0.0905	0.1061	0.1084	0.1084	0.1084	0.3015
16	0.0219	0.0354	0.0477	0.0528	0.0533	0.0533	0.0533	0.2951
	0.0368	0.0636	0.0933	0.1092	0.1105	0.1105	0.1105	0.2073
21	0.0223	0.0360	0.0484	0.0536	0.0541	0.0541	0.0541	0.2249
	0.0374	0.0647	0.0947	0.1108	0.1132	0.1132	0.1132	0.1580
41	0.0229	0.0369	0.0496	0.0548	0.0554	0.0554	0.0554	0.1152
	0.0384	0.0663	0.0970	0.1134	0.1157	0.1157	0.1157	0.0809