Techniques on Partial Fractions

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Abstract: Since graphing calculators (such as TI-89) and computer algebra systems can do Partial Fraction Decomposition (PFD), some teachers pay less attention on PFD. Although PFD is quite technical, we believe PFD also enable students to learn mathematical ideas and methods. Recently, there have been some papers discussing about PFD [1, 2, 3]. The standard method of PFD is the Method of Undetermined Coefficients (MUC), with which the calculation is often tedious. However, we can formulize the process of partial fractions, and then all calculations of partial fractions simply are related to three formulas. In addition, PFD obtained by MUC is often not ready for integration. We proposed a different format of PFD that can be for immediate integration. The techniques discussed here have not been found in textbooks, and these alternative techniques can inspire students to explore and understand partial fractions.

1. Techniques on Partial Fraction Decomposition

Technique 1: All decompositions of partial fraction can be done by applying the following three formulas:

Formula I: $\dfrac{1}{[p(x)+a][p(x)+b]} = \dfrac{1}{b-a}\left(\dfrac{1}{p(x)+a} - \dfrac{1}{p(x)+b}\right),\ a \neq b$

Formula II: $\dfrac{1}{(x-a)(x^2+bx+c)} = \dfrac{1}{f(a)}\left(\dfrac{1}{x-a} - \dfrac{x+a+b}{f(x)}\right),$

where $f(x) = x^2+bx+c$ and $f(a) \neq 0$.

Formula III: $\dfrac{1}{(x^2+\beta x+\delta)(x^2+bx+c)} = \dfrac{1}{(\beta-b)f_1(m)}\left(\dfrac{x+m+\beta}{x^2+\beta x+\delta} - \dfrac{x+m+b}{x^2+bx+c}\right),$

where $\beta \neq b$, $m = -\dfrac{\delta-c}{\beta-b}$, $f_1(x) = x^2+\beta x+\delta$, and $f_1(m) \neq 0$.

Technique 2: When we integrate a rational function, we can break the rational function into partial fractions in terms of derivatives of denominators:

If the denominator has a factor, $(ax^2+bx+c)^n$, *where* $f(x) = ax^2+bx+c$ *is not* *factorable in real numbers and* $a \neq 0$*, then we can have the following partial fractions:*

$$\frac{A_1 f'(x) + B_1}{f(x)} + \frac{A_2 f'(x) + B_2}{f^2(x)} + \ldots + \frac{A_n f'(x) + B_n}{f^n(x)}.$$

2. Proof of Formulas

Formula I. $$\frac{1}{[p(x)+a][p(x)+b]} = \frac{1}{b-a}\left(\frac{1}{p(x)+a} - \frac{1}{p(x)+b}\right),\ a \neq b.$$

Formula I was discussed in [3] by Huang and examples can be found there. The next two formulas have not been seen in any publications.

Formula II. $$\frac{1}{(x-a)(x^2+bx+c)} = \frac{1}{f(a)}\left(\frac{1}{x-a} - \frac{x+a+b}{f(x)}\right),$$

where $f(x) = x^2 + bx + c$ and $f(a) \neq 0$.

Proof of Formula II by derivation: Let

$$\frac{1}{(x-a)(x^2+bx+c)} = \frac{\alpha}{x-a} + \frac{\beta x + \gamma}{x^2+bx+c}.$$

Then the numerators satisfy

$$1 = \alpha(x^2+bx+c) + (\beta x + \gamma)(x-a).$$

Take $x = a,$ then $1 = \alpha(a^2+ab+c) = \alpha f(a),$ or $\alpha = \dfrac{1}{f(a)}.$

Take $x = 0,$ then $1 = \alpha c - a\gamma = \dfrac{1}{f(a)}c - a\gamma,$ so $\gamma = \dfrac{c}{af(a)} - \dfrac{1}{a}.$

Take $x = 1,$ then $1 = \alpha(b+c+1) + (\beta+\gamma)(1-a),$ or

$$1 = \frac{1}{f(a)}(b+c+1) + \left(\beta + \frac{c}{af(a)} - \frac{1}{a}\right)(1-a).$$

Solve for β, we get $\beta = -\dfrac{1}{f(a)}.$ Therefore

$$\frac{1}{(x-a)(x^2+bx+c)} = \frac{1}{f(a)}\frac{1}{x-a} + \frac{-\dfrac{1}{f(a)}x + \dfrac{c}{af(a)} - \dfrac{1}{a}}{x^2+bx+c}$$

$$= \frac{1}{f(a)}\left(\frac{1}{x-a} - \frac{x+a+b}{f(x)}\right).$$

Formula III. $$\frac{1}{(x^2+\beta x+\delta)(x^2+bx+c)} = \frac{1}{(\beta-b)f_1(m)}\left(\frac{x+m+\beta}{x^2+\beta x+\delta} - \frac{x+m+b}{x^2+bx+c}\right),$$

where $\beta \neq b$, $m = -\dfrac{\delta - c}{\beta - b}$, $f_1(x) = x^2 + \beta x + \delta$, and $f_1(m) \neq 0$.

Remark 2.1. Switching the positions between $f_1(x) = x^2 + \beta x + \delta$ and $f_2(x) = x^2 + bx + c$ in the above formula implies that $f_2(m) \neq 0$. We can switch the positions of $f_1(x)$ and $f_2(x)$ if evaluating $f_2(m)$ is easier. If $\beta = b$, people can apply Formula I.

Proof of Formula III by derivation. Let the fraction have the following decomposition:

$$\frac{1}{(x^2 + \beta x + \delta)(x^2 + bx + c)} = A\left(\frac{x+u}{x^2 + \beta x + \delta} - \frac{x+v}{x^2 + bx + c}\right),$$

where A, u, v are undetermined coefficients. By completing the subtraction of the right hand side, we get

$$1 = A[(b + u - \beta - v)x^2 + (c + ub - \delta - v\beta)x + uc - v\delta].$$

Therefore

$$\begin{cases} u - v + b - \beta = 0 & (1) \\ ub - v\beta + c - \delta = 0 & (2) \\ A(uc - v\delta) = 1 & (3) \end{cases}$$

Equation (3) indicates $A = \dfrac{1}{uc - v\delta}$. Multiplied by b, Equation (1) becomes

$$ub - vb + b^2 - \beta b = 0 \quad (4)$$

Equation (4) minus Equation (2), we get $v(\beta - b) + b^2 - \beta b - c + \delta = 0$, or

$$v = -\frac{b^2 - \beta b - c + \delta}{\beta - b} = b + \frac{c - \delta}{\beta - b}, \text{ and}$$

$$u = v + \beta - b = b + \frac{c-\delta}{\beta - b} + \beta - b = \beta + \frac{c - \delta}{\beta - b}.$$

$$\begin{aligned} uc - v\delta &= \left(\beta + \frac{c-\delta}{\beta - b}\right)c - \left(b + \frac{c-\delta}{\beta - b}\right)\delta \\ &= \beta c - b\delta + \frac{c-\delta}{\beta - b}(c - \delta) \\ &= \beta c - bc + bc - b\delta + \frac{(c-\delta)^2(\beta - b)}{(\beta - b)^2} \\ &= (\beta - b)\left(\frac{c-\delta}{\beta - b}\right)^2 + (\beta - b)b\frac{c-\delta}{\beta - b} + (\beta - b)c \\ &= (\beta - b) f_1(m) \end{aligned}$$

$$x + u = x + \beta + \frac{c-\delta}{\beta - b} = x + m + \beta$$

$$x + v = x + b + \frac{c-\delta}{\beta - b} = x + m + b$$

Therefore

$$\frac{1}{(x^2+\beta x+\delta)(x^2+bx+c)} = A\left(\frac{x+u}{x^2+\beta x+\delta} - \frac{x+v}{x^2+bx+c}\right)$$
$$= \frac{1}{uc - v\delta}\left(\frac{x+m+\beta}{x^2+\beta x+\delta} - \frac{x+m+b}{x^2+bx+c}\right)$$

This completes the proof of Formula III.

Proof of Formula III by Verification.

$$\frac{1}{(\beta-b)f_1(m)}\left(\frac{x+m+\beta}{x^2+\beta x+\delta} - \frac{x+m+b}{x^2+bx+c}\right)$$
$$= \frac{1}{(\beta-b)f_1(m)}\left[\frac{m(c-\delta)+\beta c - b\delta}{(x^2+\beta x+\delta)(x^2+bx+c)}\right]$$
$$= \frac{1}{(\beta-b)f_1(m)}\left[\frac{m(c-\delta)+\beta c - \beta\delta + \beta\delta - b\delta}{(x^2+\beta x+\delta)(x^2+bx+c)}\right]$$
$$= \frac{m(c-\delta)+\beta(c-\delta)+\delta(\beta-b)}{(\beta-b)f_1(m)}\frac{1}{(x^2+\beta x+\delta)(x^2+bx+c)}$$
$$= \frac{1}{f_1(m)}\left[m\left(\frac{c-\delta}{\beta-b}\right)+\beta\left(\frac{c-\delta}{\beta-b}\right)+\delta\right]\frac{1}{(x^2+\beta x+\delta)(x^2+bx+c)}$$
$$= \frac{1}{(x^2+\beta x+\delta)(x^2+bx+c)}.$$

3. Examples

Example 3.1. $$\frac{1}{(x^2-3x+4)(2x^2-x+1)} = \frac{1}{2}\frac{1}{(x^2-3x+4)(x^2-\frac{1}{2}x+\frac{1}{2})}$$

(by Formula III, let $\beta=-3$, $\delta=4$, $b=-\frac{1}{2}$, $c=\frac{1}{2}$, $m=-\frac{\delta-c}{\beta-b}=\frac{7}{5}$ and $f_1(x)=x^2-3x+4$.)

$$= \frac{1}{2}\frac{1}{-\frac{5}{2}\times\frac{44}{25}}\left[\frac{x+\frac{7}{5}+(-3)}{(x^2-3x+4)} - \frac{x+\frac{7}{5}+\left(-\frac{1}{2}\right)}{\left(x^2-\frac{1}{2}x+\frac{1}{2}\right)}\right]$$
$$= -\frac{5}{44}\left(\frac{x-\frac{8}{5}}{x^2-3x+4} - \frac{x+\frac{9}{10}}{x^2-\frac{1}{2}x+\frac{1}{2}}\right)$$

Remark 3.2. Partial fraction decomposition is considered complete if it only contains expressions like $\dfrac{A}{(x-a)^m}$ and $\dfrac{Bx+C}{(x^2+bx+c)^n}$, where *m* and *n* are positive integers and x^2+bx+c is prime in real numbers.

Example 3.2.

$$\frac{3x^2+4x+4}{x^3+4x}=\frac{3x^2+4x+4}{x(x^2+4)} \qquad \text{(factoring the denominator)}$$

$$=\frac{3(x^2+4)-12+4x+4}{x(x^2+4)} \qquad (\textit{plus}\ 12 \text{ and } \textit{minus}\ 12)$$

$$=\frac{3}{x}+\frac{4}{x^2+4}-\frac{8}{x(x^2+4)} \qquad \text{(decomposition)}$$

$$=\frac{3}{x}+\frac{4}{x^2+4}-2\left(\frac{1}{x}-\frac{x}{x^2+4}\right) \qquad \text{(by Formula III)}$$

$$=\frac{1}{x}+\frac{2x+4}{x^2+4}$$

Example 3.3.

$$\frac{x}{(x+1)(x^2+x+2)^2}=\frac{x+1-1}{(x+1)(x^2+x+2)^2} \qquad (\textit{plus}\ 1 \text{ and } \textit{minus}\ 1)$$

$$=\frac{1}{(x^2+x+2)^2}-\frac{1}{(x^2+x+2)}\,\frac{1}{(x+1)(x^2+x+2)} \qquad \text{(separating the fractions)}$$

$$=\frac{1}{(x^2+x+2)^2}-\frac{1}{(x^2+x+2)}\,\frac{1}{2}\left(\frac{1}{x+1}-\frac{x-1+1}{x^2+x+2}\right) \qquad \text{(by Formula II)}$$

$$=\frac{1}{(x^2+x+2)^2}-\frac{1}{2}\,\frac{1}{(x+1)(x^2+x+2)}+\frac{1}{2}\,\frac{x}{(x^2+x+2)^2}$$

$$=\frac{1}{(x^2+x+2)^2}-\frac{1}{2}\,\frac{1}{2}\left(\frac{1}{x+1}-\frac{x-1+1}{x^2+x+2}\right)+\frac{1}{2}\,\frac{x}{(x^2+x+2)^2} \qquad \text{(by Formula II)}$$

$$=-\frac{1}{4}\left(\frac{1}{x+1}\right)+\frac{1}{4}\left(\frac{x}{x^2+x+2}\right)+\frac{1}{(x^2+x+2)^2}+\frac{1}{2}\,\frac{x}{(x^2+x+2)^2}\,.$$

4. Integration by Partial Fractions in Terms of the Derivative of a Factor

The common format of partial fractions with a quadratic factor, $(ax^2+bx+c)^n$, which is not factorable in real numbers, is

$$\frac{A_1x+B_1}{ax^2+bx+c}+\frac{A_2x+B_2}{(ax^2+bx+c)^2}+\ldots+\frac{A_nx+B_n}{(ax^2+bx+c)^n}\,. \qquad (1)$$

But integrating a rational function with this format, students can still be frustrated because it is not ready for integration. For example, by decomposing the rational function,

$$\int\frac{x-1}{(x^2+2x+2)(x+2)}dx=\int\frac{\frac{3}{2}x+1}{x^2+2x+2}-\frac{\frac{3}{2}}{x+2}\,dx\,. \qquad (2)$$

Many students still have difficulties to finish $\int \frac{\frac{3}{2}x+1}{x^2+2x+2}dx$. Almost all calculus textbooks use this format of partial fractions. To make the decomposition ready for integration by substitution, I taught students with an alternative format:

If the denominator has a factor, $(ax^2+bx+c)^n$*, where* $f(x)=ax^2+bx+c$ *is not factorable in real numbers and* $a \neq 0$*, then we can have the following partial fractions:*

$$\frac{A_1 f'(x)+B_1}{f(x)}+\frac{A_2 f'(x)+B_2}{f^2(x)}+\ldots+\frac{A_n f'(x)+B_n}{f^n(x)}. \tag{3}$$

Formats (1) and (3) are equivalent. But (3) has the following advantages:

◊ It is ready for integration by substitution.

◊ Finding undetermined coefficients of (3) is easier than (1) because we can use the zero of $f'(x)$.

Let us revisit Equation (2): find $\int \frac{x-1}{(x^2+2x+2)(x+2)}dx$.

Solution:

$$\frac{x-1}{(x^2+2x+2)(x+2)}=\frac{A(2x+2)+B}{x^2+2x+2}+\frac{C}{x+2}$$

$$x-1=[A(2x+2)+B](x+2)+C(x^2+2x+x)$$

Let $x=-2, -3=2C, C=-\frac{3}{2}$.

Let $x=-1$ (the zero of $f'(x)$), $-2=B+C$, or $-2=B-\frac{3}{2}$, and $B=-\frac{1}{2}$.

Let $x=0$, $-1=2(2A+B)+2C$, $A=\frac{3}{4}$.

Thus,

$$\int \frac{x-1}{(x^2+2x+2)(x+2)}dx=\int\left(\frac{3}{4}\cdot\frac{2x+2}{x^2+2x+2}-\frac{1}{2}\cdot\frac{1}{x^2+2x+2}-\frac{3}{2}\cdot\frac{1}{x+2}\right)dx,$$

and the integral can be finished more effectively than (2).

Reference:

1. Brazier, Richard and Boman, Eugene (2007), *How to Compute the Partial Fraction Decomposition Without Really Trying*, The AMATYC Review, Vol. 29, No.1, 20-29.
2. Cherif, Chokri (2007), *An Alternative Method to the Classical Partial Fraction Decomposition*, The AMATYC Review, Vol. 29, No.1, 31-37.
3. Huang, XunCheng (1991), *A Shortcut in Partial Fractions*, College Mathematics Journal, Vol. 22, No.5, 413—415.