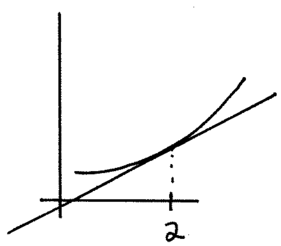
Definition of the Derivative.

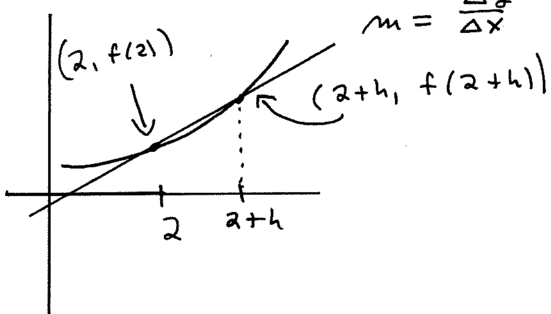
Find the slope of the tangent to the graph of $y = f(x)$ at the point where $x = 2$.



Questions?

1. What is a tangent?
2. How can we compute slope?
3. Can we find approximate solution?

$m = \frac{\Delta y}{\Delta x}$



$$m_{secant} = \frac{\Delta y}{\Delta x} = \frac{f(2+h) - f(2)}{(2+h) - 2} = \frac{f(2+h) - f(2)}{h}$$

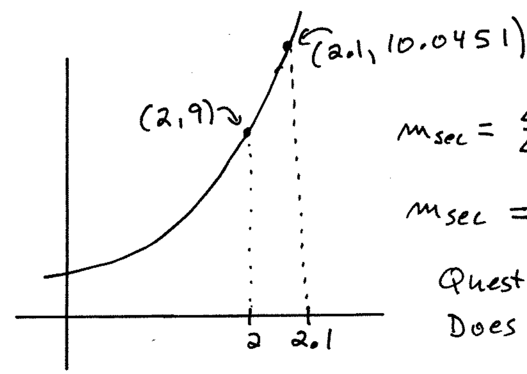
$$m_{tangent} = \lim_{h \to 0} m_{secant} = \lim_{h \to 0} \frac{f(2+h) - f(2)}{h}$$

Notation:

$f'(2)$ means the slope of the tangent to the graph of $y = f(x)$ at the point where $x = 2$.

So $f'(2) = m_{tangent} = \lim_{h \to 0} \frac{f(2+h) - f(2)}{h}$

Example: Let $f(x) = 3^x$. Approximate $f'(2)$



$m_{sec} = \frac{\Delta y}{\Delta x} = \frac{10.0451 - 9}{.1}$

$m_{sec} = 10.45$

Question:
Does $f'(2) = 10.45$?

Question: What does $f'(x)$ mean?

Question: What type of graph would not have the slope of a tangent at $x = 2$?

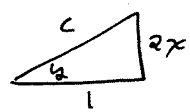
Derivatives of Inverse Functions.

Let $f(x) = \tan^{-1} 2x$. Find $f'(x)$.

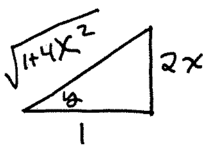
$y = \tan^{-1} 2x$. Goal: Find $\frac{dy}{dx}$

If $y = \tan^{-1} 2x$, then $\tan y = 2x$

But $\tan \theta = \frac{\text{Opp}}{\text{Adj}}$ so



$1^2 + (2x)^2 = c^2 \Rightarrow 1 + 4x^2 = c^2 \Rightarrow \sqrt{1+4x^2} = c$



$\tan y = 2x$

$\frac{d}{dx} \tan y = \frac{d}{dx} 2x$

$\sec^2 y \frac{dy}{dx} = 2$

$\frac{dy}{dx} = \frac{2}{\sec^2 y}$

$\frac{dy}{dx} = \frac{2}{1+4x^2}$

But $\cos y = \frac{\text{ADJ}}{\text{HYP}}$

$\cos y = \frac{1}{\sqrt{1+4x^2}}$

$\sec y = \sqrt{1+4x^2}$

$\sec^2 y = 1 + 4x^2$

So when $f(x) = \tan^{-1} 2x$

$f'(x) = \frac{2}{1+4x^2}$

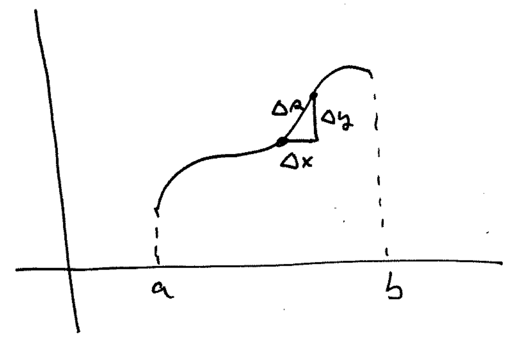
Let $y = f^{-1}(x)$. Find y'.

$f(y) = x \Rightarrow \frac{d}{dx} f(y) = x \Rightarrow f'(y) \frac{dy}{dx} = 1$

$\Rightarrow \frac{dy}{dx} = \frac{1}{f'(y)} \Rightarrow \frac{dy}{dx} = \frac{1}{f'(f^{-1}(x))}$

Arc-Length

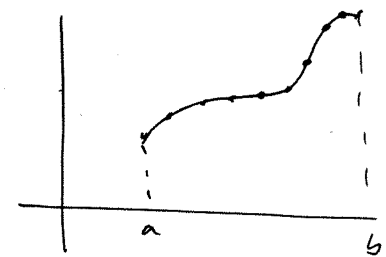
Find the arc length of $y = f(x)$ on $[a,b]$



From Pythagoras we know $(\Delta x)^2 + (\Delta y)^2 = (\Delta \ell)^2$

So $\Delta \ell = \sqrt{(\Delta x)^2 + (\Delta y)^2}$

Now break up the graph into many small parts



So
Total arc length =

$\sum \Delta \ell$

$\ell = \sum \sqrt{(\Delta x)^2 + (\Delta y)^2}$

$\ell = \sum \sqrt{\frac{(\Delta x)^2}{(\Delta x)^2} + \frac{(\Delta y)^2}{(\Delta x)^2}}\ \Delta x = \sum \sqrt{1 + \left(\frac{\Delta y}{\Delta x}\right)^2}\ \Delta x$

Shrink Δx and Δy to 0.

$\ell = \int \sqrt{1 + (dy/dx)^2}\ dx$

Taylor and Maclaurin Series

Find a Maclaurin series for $f(x) = e^{2x}$

Goal: Find a polynomial $p(x)$ that approximates $f(x) = e^{2x}$.

$p(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + \cdots$

$f(x) = e^{2x}$

we want $p(0) = f(0)$, $p'(0) = f'(0)$, $p''(0) = f''(0)$ etc.

$p(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + \cdots$

$p'(x) = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + 5a_5 x^4 + \cdots$

$p''(x) = 2a_2 + 6a_3 x + 12 a_4 x^2 + 20 a_5 x^3 + \cdots$

$p'''(x) = 6a_3 + 24 a_4 x + 60 a_5 x^2 + \cdots$

$p''''(x) = 24 a_4 + 120 a_5 x + \cdots$

$p^{(v)}(x) = 120 a_5 + \_\_ x + \_\_ x^2 + \_\_ x^3 + \cdots$

$p(0) = a_0$ $\quad p'(0) = a_1$ $\quad p''(0) = 2a_2$ $\quad p'''(0) = 6a_3$

$p''''(0) = 24 a_4$ $\quad p^{(v)}(0) = 120 a_5$ $\quad p^{(vi)}(0) = ?$

$f(x) = e^{2x}$	$f(0) = 1$
$f'(x) = 2e^{2x}$	$f'(0) = 2$
$f''(x) = 4e^{2x}$	$f''(0) = 4$
$f'''(x) = 8e^{2x}$	$f'''(0) = 8$
$f''''(x) = 16e^{2x}$	$f''''(0) = 16$
$f^{(v)}(x) = 32e^{2x}$	$f^{(v)}(0) = 32$

$p(0) = f(0)$

$a_0 = 1$

$p'(0) = f'(0)$

$a_1 = 2$

$p''(0) = f''(0)$

$2a_2 = 4$

$a_2 = 2$

$p'''(0) = f'''(0)$

$6a_3 = 8$

$a_3 = \frac{8}{6} = \frac{4}{3}$

$p''''(0) = f''''(0)$

$24 a_4 = 16$

$a_4 = \frac{16}{24} = \frac{2}{3}$

$p^{(v)}(0) = f^{(v)}(0)$

$120 a_5 = 32$

$a_5 = \frac{32}{120} = \frac{4}{15}$

$p^{(vi)}(0) = f^{(vi)}(0)$

$6!\, a_6 = 64$

$a_6 = \frac{64}{6!} = \frac{4}{45}$

So $f(x) = e^{2x} \approx 1 + 2x + 2x^2 + \frac{4}{3}x^3 + \frac{2}{3}x^4 + \frac{4}{15}x^5 + \frac{4}{45}x^6 + \cdots$

Is there a pattern?

$p^{(N)}(0) = N!\, a_N$

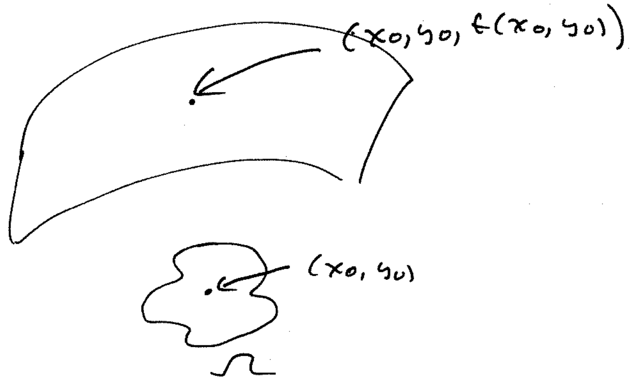
we need $p^{(N)}(0) = f^{(N)}(0)$ so

$N!\, a_N = f^{(N)}(0)$

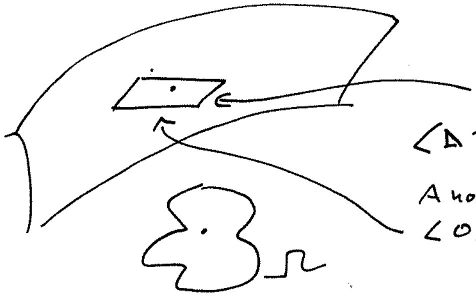
$$a_N = \frac{f^{(N)}(0)}{N!}$$

Surface Area

Find the area of the surface $z = f(x,y)$ above the region Ω.



"Glue" a small piece of "papier-mache" that represents the tangent plane to (x_0, y_0) at the point $(x_0, y_0, f(x_0, y_0))$.



one vector is

$\langle \Delta x, 0, f_x \Delta x \rangle = u$

Another vector is

$\langle 0, \Delta y, f_y \Delta y \rangle = v$

What is the area of that piece of papier-mache?

Answer: $\|u \times v\|$

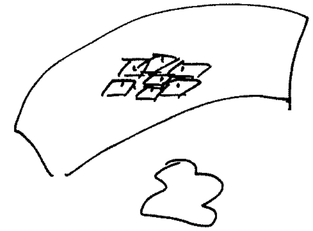
$$u \times v = \det \begin{bmatrix} i & j & k \\ \Delta x & 0 & f_x \Delta x \\ 0 & \Delta y & f_y \Delta y \end{bmatrix}$$

$$u \times v = \langle -f_x \Delta x \Delta y, -f_y \Delta x \Delta y, \Delta x \Delta y \rangle$$

$$= |\Delta x \Delta y| \langle -f_x, -f_y, 1 \rangle$$

$$\|u \times v\| = \sqrt{f_x^2 + f_y^2 + 1} \quad \Delta x \Delta y$$

The total surface area is the sum of all the pieces of papier mache over the region Ω.



$$SA \cong \sum \sqrt{f_x^2 + f_y^2 + 1} \quad \Delta x \Delta y$$

Shrink Δx and Δy to zero, and get

$$SA = \iint_{\Omega} \sqrt{f_x^2 + f_y^2 + 1} \; dx\, dy$$