

The French Mathematical Impact: Descartes, Fermat, and Beyond

David and Elise Price

Tarrant County College, Southeast Campus

Arlington, Texas

mathprofs@sbcglobal.net

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France has contributed to the development of mathematics for more than a thousand years. Because the French founded New Orleans, we thought that it would be interesting to examine some of these achievements. While France officially became a nation in 987 with the ascension of Hugh Capet to the throne, let us begin with Charlemagne (742-814) in the eighth century. Although illiterate himself, he recognized the need for education, and he launched the Carolingian Renaissance about 780 by inviting scholars, including the Englishman, Alcuin of York (ca. 735-804), to come to his court. The most accomplished intellectual of his time, Alcuin introduced a curriculum based on the seven liberal arts of the quadrivium (arithmetic, geometry, astronomy, and music) and the trivium (logic, grammar, and rhetoric). He wrote elementary texts in each of these areas, searched for additional manuscripts to be read by his students, and is thought to have composed a book entitled *Propositions for Sharpening Youthful Minds*, which consisted of fifty-three logical and arithmetic puzzles.

Educational progress was slow in Charlemagne's empire, which began to disintegrate shortly after his death because of barbarian invasions. In the tenth century, Gerbert (ca. 940-1003) was born in France and eventually became Pope Sylvester II, the first French pope. He developed an interest in mathematics and wrote on arithmetic and geometry. His most significant mathematical achievement was the introduction of the Hindu-Arabic number system into Europe although he may not have understood that zero can be treated as a number.

Roger Bacon (1214-1292), like Alcuin, was English, but he taught at the University of Paris and was known as the "wonderful teacher." Although he was more interested in science than mathematics, he understood the value of quantification in the pursuit of scientific knowledge. His writings contain such statements as:

Neglect of mathematics works injury to all knowledge, since he who is ignorant of it cannot know the other sciences or the things of this world. And what is worse, men who are thus ignorant are unable to perceive their own ignorance and so do not seek a remedy;

If in the other sciences we should arrive at certainty without doubt and truth without error, it behooves us to place the foundations of knowledge in mathematics;

Mathematics is the gate and key of the sciences.

Among his achievements, Bacon was the first person to advocate reform of the Julian calendar. In one of his most revolutionary insights, he sometimes hinted at the use of experimentation in science, an idea that was several centuries ahead of its time. He also grasped the possibility of applying scientific discoveries to invention. There are suggestions in his writings of submarines, automobiles, and airplanes.

Nicole Oresme (ca. 1323-1382), who was born in Normandy and attended the University of Paris, contributed to the advancement of algebra by introducing rational exponents and the rules for computing with them. He also considered the possibility of irrational exponents although he lacked the notation necessary to pursue the idea. While studying the concept of uniform acceleration, he decided to display the relationship between time and velocity graphically, a technique that eventually led to the geometric representation of functions. The ancient Greek geometer, Apollonius, in his monumental work on conic sections, associated curves with equations, but he did not associate equations with curves. By establishing this latter connection, Oresme's work helped to prepare the way for the creation of analytic geometry in the seventeenth century.

In order to calculate the distance traveled by an object subject to non-uniform velocity, Oresme investigated infinite series. Although the explicit formulation of the concept of a limit lay several centuries in the future, his work stimulated interest in the ideas of convergence and divergence. He gave the first proof that the harmonic series diverges, a demonstration that appears in calculus textbooks to this day.

Nicolas Chuquet (1445-1500), the best French mathematician of the next century, introduced negative numbers and zero as exponents, a step that represented the first use of negative numbers in Europe. He also dealt with negative coefficients and in some instances, negative solutions of equations although he was not consistent in this last respect. His book, *Triparty*, written in 1484, was the earliest work on Renaissance algebra, especially the solution of equations. As the name implies, it consisted of three parts. The other two dealt with the Hindu-Arabic number system and the calculation of roots.

The leading French mathematician of the sixteenth century was François Viète (1540-1603), one of whose innovations was the use of decimal fractions. Earlier advocates of the Hindu-Arabic number system had used it for natural numbers only and had continued to express fractional parts in sexagesimal form. Viète completed the transition to a full place-value decimal system.

Viète's most original contributions were to algebra. Although Italian mathematicians showed how to solve cubic and quartic equations by radicals, they did so by means of numerical examples, not by writing the general equations symbolically. Indeed, while the quadratic formula had been known since the time of the ancient Babylonians, the form in which we express it today had not been devised. Instead of illustrating problem solving techniques by means of specific

demonstrations, Viète inaugurated a major advancement in algebraic notation by using vowels to represent unknowns and consonants to represent known values. He was then able to inaugurate the study of the theory of equations by proving relationships among the roots of an equation of a given type rather than focusing merely on the solutions of a specific equation. In the next century, Descartes used letters toward the end of the alphabet for variables and ones toward the beginning for constants, a practice still common today.

Before trigonometry became a separate branch of mathematics, Euclid stated and proved the law of cosines in purely geometric terms in the *Elements*. Viète formulated the trigonometric version that we use today, and he was also the first mathematician to state the law of tangents. His interest in the analytic aspects of the subject led him to develop the sum to product identities that we use for $\sin x + \sin y$ and $\cos x + \cos y$, as well as the formulas for $\sin nx$ and $\cos nx$.

Prior to the seventeenth century, geometry and algebra had developed as independent branches of mathematics. The creation of analytic geometry by René Descartes (1596-1650) and Pierre de Fermat (1601-1665) integrated the two and prepared the way for calculus. Descartes' primary goal was to solve geometric construction problems by converting them to algebraic equations through the introduction of coordinates. His system was essentially the same as the one we use although his two axes were not perpendicular to each other, and he did not explicitly include the equivalent of our y-axis in his diagrams. He did use the letters x and y to denote the abscissa and the ordinate respectively. His work significantly extended the concept of a curve beyond the lines, circles, and other conic sections studied previously.

Prior to the seventeenth century, mathematicians had not focused on finding tangents to curves. Descartes developed a method of finding a normal to a curve at a point, from which, of course, he could then calculate the slope of the tangent there. He also contributed to the theory of equations. For example, he observed that a polynomial in x is divisible by $x - a$ if and only if a is root of the polynomial. He gave an intuitive proof of the Fundamental Theorem of Algebra, later proved much more rigorously by Gauss, and he devised his well-known rule of signs. Another of his innovations was the recognition of negative and complex roots.

Whereas Descartes' development of analytic geometry began with a curve and moved to its equation, Fermat started with an equation and determined the corresponding curve. His fundamental goal was to express Apollonius' results in algebraic form. By examining linear and quadratic equations, he graphed lines and conic sections. His coordinate system corresponded to the one we use in that his two axes were generally perpendicular rather than oblique.

Fermat used his achievements in analytic geometry to take steps toward the development of calculus. In a treatise entitled *Method of Finding Maxima and Minima*, he essentially took the derivative of a polynomial function and set it to zero to find its

relative extrema. By extending his technique, he calculated the slope of a curve at any point on it. He did not have the explicit concept of a limit, but otherwise his method is the one we use today. He also approximated the area under a curve by means of circumscribed rectangles. Although he left no record that he recognized the inverse relationship between differentiation and integration, his work strongly hinted at the Fundamental Theorem of Calculus.

Of course, Fermat is most famous for his work in number theory, including his Last Theorem. He also collaborated with Blaise Pascal (1623-1662) to found the subject of probability.

One of the most original mathematical developments of the Renaissance was the study of perspective, motivated by Italian painters who wanted to develop techniques of representing three-dimensional objects on a two-dimensional surface. Several artists contributed to this undertaking, including Leon Battista Alberti (1404-1472), who wrote the first book on the subject, in which he commented that the most fundamental requirement of a painter is to know geometry, and Piero della Francesca (1420-1492), who advanced Alberti's investigation still further in a tract in which he thoroughly discussed the mathematical basis of painting. Girard Desargues (1591-1661), a French engineer and architect, gave the subject its first formal treatment by launching the field of projective geometry.

Desargues' contribution generated little interest initially because it was quite difficult to read. One of his admirers, however, was Pascal, who contributed an original theorem to projective geometry. He also invented the first automated calculating device, one that would add and subtract, and as we have mentioned, he founded the study of probability with Fermat. Of course, his name is associated with the triangle that generates the binomial coefficients although the triangle had been known since the twelfth century. However, Pascal investigated its properties in great depth and in the process explicitly stated and used the principle of mathematical induction.

When we hear the name of Abraham de Moivre (1667-1754), we naturally think of his famous theorem and the calculation of the n th roots of a real or complex number. However, he did significant work in probability as well, addressing actuarial problems and questions related to life annuities. Additional contributions included advancing the understanding of infinite series and devising what is known today as Stirling's formula to approximate $n!$ for large values of n .

In 1692, Johann Bernoulli tutored a French marquis, G. F. A. de L'Hôpital (1661-1704), in calculus, and they signed an agreement in which Bernoulli granted L'Hôpital use of his mathematical discoveries in exchange for a regular salary. In 1696, L'Hôpital published the first textbook on differential calculus, and it included one of Bernoulli's major results, the technique that we call L'Hôpital's Rule. The book was well written and enjoyed considerable success throughout the eighteenth century, as did another by L'Hôpital on analytic geometry.

Beginning with the work of Newton and Leibniz, mathematicians had struggled to identify a defensible conceptual foundation for calculus. Jean Le Rond d'Alembert (1717-1783) suggested that the notion of a limit should be taken as the fundamental idea, and he attempted to define it. His definition lacked precision in terms of inequalities, but he helped to prepare the way for the rigorous development of the subject in the nineteenth century. He also made an extensive effort to prove the Fundamental Theorem of Algebra. Although his proof was incomplete, the theorem is known in France today as d'Alembert's theorem. Among his other achievements were formulating the ratio test for absolute convergence of an infinite series, identifying what we call the Cauchy-Riemann conditions to determine if a complex function is analytic, and being one of the pioneers in the field of partial differential equations.

Emilie du Châtelet (1706-1749) was a precocious child who developed fluency in multiple languages by the age of twelve. Her strongest interest, however, was mathematics, and she is best remembered for her translation of and commentary on Newton's *Principia*. It helped to convince France that Newton's explanation of the solar system was superior to Descartes'.

Joseph-Louis Lagrange (1736-1813) was one of the leading mathematicians of the eighteenth century. Born in Italy to an Italian father and a French mother, he spent a significant portion of his career in Paris. His *Analytical Mechanics* gave an axiomatic treatment of Newtonian mechanics, and although his attempt to develop calculus on the basis of infinite series was not successful, in the process he launched the study of the theory of functions of a real variable. As part of this effort, he introduced the term "derived function," from which our word "derivative" comes, and he devised the prime notation for derivatives. Among his other accomplishments were the first formulation of the Mean Value Theorem, the technique of Lagrange multipliers, and the method of variation of parameters to solve nonhomogeneous, linear differential equations.

In algebra, Lagrange's work on the theory of equations prepared the way for the concept of a group. Of course, the theorem that the order of a subgroup of a finite group divides the order of the group is named for him. In number theory, he studied congruences, and he proved that every positive integer is the sum of at most four perfect squares.

Nicolas de Condorcet (1743-1794) exhibited the strong interest of numerous mathematicians during the eighteenth century in the application of mathematics to social questions, as well as the physical sciences. In addition to publishing books on both probability and integral calculus, he developed the pairwise comparison voting method that bears his name. He was also a member of the French Committee on Weights and Measures, which established the metric system, as was Lagrange. When some groups in France opposed inoculation for smallpox, Condorcet used his knowledge of probability and statistics to defend the practice.

For centuries the study of solid geometry had been subordinated to plane geometry. Plato criticized the neglect of the subject and encouraged mathematicians at his Academy to

pursue it. Gaspard Monge (1746-1818) developed solid analytic geometry and applied calculus to the study of curves and surfaces in three dimensions. We teach much of his work in our calculus sequence today. He also introduced descriptive geometry by showing how to project a figure onto two orthogonal planes, a technique that is fundamental to engineering design. Monge was also famous as an excellent teacher and administrator, and he was active in both capacities as one of the founders of the École Polytechnique. In addition, he too was a member of the Committee on Weights and Measures.

Pierre Simon de Laplace (1749-1826) made major contributions both to mathematics and mathematical physics. His work on the theory of probability was more significant than that of any other mathematician, and by completing the gravitational portion of Newton's work in his monumental *Celestial Mechanics*, he showed that the solar system is stable. He also improved Newton's calculation of the speed of sound. Of course, he was responsible for the method of Laplace transforms in differential equations.

Confined to her house during the violence of the French Revolution, Sophie Germain (1776-1831) taught herself calculus, using books from her father's library. As a woman, she was barred from admittance to the École Polytechnique, but she was allowed to obtain lecture notes of professors who taught there and to submit written comments. Her work, submitted under the pseudonym of Monsieur Antoine-August Le Blanc, a student at the school, favorably impressed Lagrange, who was amazed to learn that she was female and who introduced her to French mathematical and scientific society. She won a prize from the French Academy for a paper on elasticity and later developed an interest in number theory after studying Gauss' work on the subject. Eventually she made a contribution to the study of Fermat's Last Theorem, and she corresponded with Gauss, again using her pseudonym. Gauss was also surprised to learn that she was a woman. He recommended that she be granted an honorary doctorate from the University of Göttingen, but she died before she could receive it.

In 1826, the Norwegian mathematician, Niels Abel, proved that the general polynomial equation of degree greater than four cannot be solved by radicals. Évariste Galois (1811-1832) then established conditions under which such a solution is possible. He explicitly introduced the concept of a permutation group, and he contributed to the study of finite fields. Although Galois Theory is not necessary to solve the three famous construction problems of ancient Greek mathematics, trisecting an angle, doubling a cube, and squaring a circle using only a straightedge and a compass, it provides a means of doing so. Galois' work helped to shift the focus of algebra from computation to investigation of the structural features of an algebraic system.

The mathematician most responsible for shaping calculus as we know it today was Augustin-Louis Cauchy (1789-1857). He understood that the concept of a limit provided the proper basis for the subject, and he defined it rigorously. He also gave definitions of continuity and differentiability, and he defined the differential in terms of the derivative, as we do. Newton thought of a derivative as an "ultimate ratio," and Leibniz viewed it as the quotient of two differentials. Cauchy was the first to conceive of it as the limit of a

quotient, not as a quotient itself, and then to show that it can be represented as the quotient of two differentials.

Although Lagrange stated the Mean Value Theorem, Cauchy used his definition of the derivative to give the first proof of it. He also proved that if a function has a positive derivative on an interval, it is increasing there, if the derivative is negative, it is decreasing, and if the derivative is zero, it is constant.

Prior to Cauchy, most mathematicians thought of integration as the inverse of differentiation rather than as the result of a process of summation. Cauchy focused on the definite integral and emphasized that it is the limit of a sum. He may have been motivated by the fact that there is no way known to antidifferentiate some functions. After proving the Mean Value Theorem for Integrals, he then established the exact relationship between definite and indefinite integration by providing the first proof of the Fundamental Theorem of Calculus. His demonstration assumed that the integrand is continuous; he later extended the concept to include an integrand with finitely many discontinuities on an interval, and he defined both types of improper integrals.

Cauchy drew a clear distinction between convergent and divergent series, and he showed that convergence is a limiting process. He developed several convergence criteria, including the comparison, ratio, and root tests and the alternating series test. He also devised what we call the Cauchy condition for convergence of a sequence. His proof of it was incomplete, however, because he lacked an adequate definition of a real number. He was able to show that the criterion is necessary but not that it is sufficient. Later in the nineteenth century, several mathematicians, including the Germans, Richard Dedekind and Georg Cantor, and the Frenchman, Charles Meray (1835-1911), resolved this issue.

Cauchy thought he had proved that a convergent series of continuous functions has a continuous limit. However, based on Joseph Fourier's (1768-1830) investigation of what we call Fourier series, Abel discovered a counterexample to Cauchy's result. In addition, because of Fourier's work, mathematicians gradually generalized the definition of a function until it became the one we use today.

Fourier's goal was to use trigonometric series to develop a mathematical theory of heat. His writing was criticized by some of his contemporaries, including Lagrange, for its lack of precision, and it was one of the developments that led nineteenth century mathematicians to focus on rigor. One example of this effort was the introduction of the concept of uniform convergence by the German mathematician, Karl Weierstrass, to correct the error in Cauchy's proof concerning the continuous limit of a convergent series of continuous functions.

Similarly, the German mathematician, Eduard Heine, introduced the concept of uniform continuity and showed that a continuous function on a closed and bounded interval is uniformly continuous there. His proof used without justification what we call the Heine-Borel covering theorem, that an open cover of a closed and bounded set has a finite

subcover. Emile Borel (1871-1956) eventually proved it on the assumption that the covering set is countable. He also contributed to the development of measure theory and the summability of series.

Henri Lebesgue (1875-1941) was a student of Borel, who advanced his teacher's work on measure theory and defined what we call the Lebesgue integral, which generalized the Riemann integral. Lebesgue also proved the Heine-Borel theorem for any open cover.

Henri Poincare (1854-1912) is considered to be the last universalist in mathematics. He wrote prolifically on a vast number of topics, including differential equations, probability, non-Euclidean geometry, celestial mechanics, quantum theory, and relativity. He was one of the founders of topology, especially algebraic topology. In his capacity as a professor of mathematical physics at the Sorbonne, he lectured each year on a different topic, such as elasticity, thermodynamics, optics, or electricity. He also wrote several nontechnical books on mathematics and science for the general public.

Beginning in 1935, Jean Dieudonné (1906-1992) and André Weil (1906-1998) led a group of mathematicians, who were primarily French, to undertake the task of developing an improved text on mathematical analysis. They soon broadened their goal to create a multi-volume work entitled *Elements of Mathematics*, which included a development of topics such as set theory, algebra, general topology, functions of one real variable, topological vector spaces, and integration. Written using the pseudonym of Nicolas Bourbaki, their work had a strong influence on mathematics and mathematics education during the middle of the twentieth century, including the development of the "new math" curriculum. It has since been criticized as being too abstract in its approach and too narrow in its scope.

During the second half of the twentieth century, French mathematicians continued to contribute to mathematics. For example, René Thom (1923-2002) developed catastrophe theory, which analyzes how continuous changes can lead to a discontinuous result. His approach uses the concept of a conservative vector field, an idea that we teach in third semester calculus.

Gaston Julia (1893-1978) and Paul Lévy (1886-1971) both taught at the École Polytechnique, where they introduced Benoit Mandelbrot (1924-2010) to the concept that he would later call a fractal. Mandelbrot was born in Poland but immigrated to France as a child with his parents to escape political persecution. He investigated underlying order in apparently chaotic processes, and his work is now included in liberal arts mathematics courses. After a distinguished career at IBM, he taught at Yale and at the age of seventy-five, became the oldest professor in its history to receive tenure.

We would like to conclude by noting some of the many connections between American and French history. During the seventeenth century the French explorer, Robert de La Salle, claimed the entire Mississippi River basin for France, and in 1718 Jean-Baptiste de Bienville founded the city of New Orleans. Following France's defeat in the Seven

Years' War, New Orleans and the French colony west of the Mississippi were ceded to Spain. In 1788 and 1795, two major fires destroyed more than a thousand buildings in the city. During the subsequent rebuilding, much of the French Quarter as we know it today was constructed, including the St. Louis Cathedral, the Cabildo, and the Presbytere.

In 1795, Spain gave Americans access to the port of New Orleans by granting the United States "Right of Deposit." By means of the Treaty of Ildefonso in 1800, Spain transferred Louisiana back to France, and three years later Napoleon sold it to the United States as the Louisiana Purchase. The Lewis and Clark expedition expanded knowledge of the continent, both geographically and scientifically.

Some of the most significant relationships between the United States and France have been forged during wartime. French assistance was crucial to our victory in the Revolutionary War just as American support was indispensable to the liberation of France during World War II. French scientists worked on the Manhattan Project, and the French mathematician, Jean Leray (1906-1998), while a Nazi prisoner of war in Austria from 1940 to 1945, concealed his expertise in differential equations so that he would not be forced to work on military projects against his will.

Several years ago, we visited the Yorktown Victory Center in Virginia, where the climatic battle of the American Revolution was fought. Prior to taking a driving tour of the area, we attended an orientation lecture, which gave an excellent overview of what we would see. At one point the presenter stated that we would stop at a cemetery in which French soldiers killed in the battle are buried. His tone then became quite intense as he pointed out that each of the French graves faces east toward France, just as each of the graves in the American cemetery at Normandy faces west toward the United States. Later that year, we had the opportunity to visit Normandy. Seeing those two sites was dramatic confirmation of the historical relationship between the two countries, both mathematical and otherwise.

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