The Prime Coefficient Formulas for Rotation of Conics

It is known that the graph of a conic section in the x, y-coordinate plane has a second degree equation in the variables x and y. I.e., $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$. It is also known that rotation of the axes through an angle about the origin, transforms the equation to $A'x^2 + B'xy + C'y^2 + D'x + E'y + F' = 0$. And, if $B \neq 0$, then rotation of the axes through an angle $\theta = \frac{1}{2}\cot^{-1}\left(\frac{A-C}{B}\right)$ yields $B' = 0$ so that in this rotated system the conic has vertical and horizontal axes of symmetry unless it is a parabola in which case the axis of symmetry is either vertical or horizontal. The rotation of axes coordinate transformations are as follows.

$$x = x'\cos\theta - y'\sin\theta$$
$$y = x'\sin\theta + y'\cos\theta$$

By substitution into $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$ we obtain $A'x^2 + B'xy + C'y^2 + D'x + E'y + F' = 0$ with the following prime coefficients.

$$A' = A\cos^2\theta + \frac{B}{2}\sin(2\theta) + C\sin^2\theta$$

$$B' = B\cos(2\theta) - (A-C)\sin(2\theta)$$

$$C' = A\sin^2\theta - \frac{B}{2}\sin(2\theta) + C\cos^2\theta$$

$$D' = D\cos\theta + E\sin\theta$$

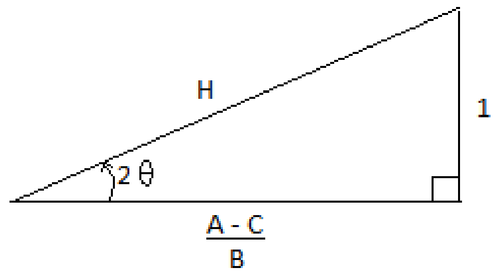
$$E' = E\cos\theta - D\sin\theta$$

$$F' = F$$

Of course, $B' = 0$ if $\theta = \frac{1}{2}\cot^{-1}\left(\frac{A-C}{B}\right)$. Then, $0 < 2\theta < \pi$ and $\sin\theta > 0$ and $\cos\theta > 0$ since $0 < \theta < \frac{\pi}{2}$. And, whether $\frac{A-C}{B} < 0$ or $\frac{A-C}{B} \geq 0$, we have

$$\cos(2\theta) = \frac{A-C}{BH} \text{ and } \sin(2\theta) = \frac{1}{H} \text{ where } H = \sqrt{\left(\frac{A-C}{B}\right)^2 + 1}.$$

This is made memorable by the right triangle for the case when 2θ is acute.



Therefore, $\cos^2\theta = \frac{1}{2}(1+\cos(2\theta)) = \frac{(A-C)+BH}{2BH}$ and $\sin^2\theta = \frac{1}{2}(1-\cos(2\theta)) = \frac{BH-(A-C)}{2BH}$. Since $\sin\theta$ and $\cos\theta$ are positive, we have

$$\sin\theta = \sqrt{\frac{BH-(A-C)}{2BH}} \text{ and } \cos\theta = \sqrt{\frac{(A-C)+BH}{2BH}}.$$

We may now formulate the prime coefficients in terms of the unprimed coefficients when $\theta = \frac{1}{2}\cot^{-1}\left(\frac{A-C}{B}\right)$. For example,

$$\begin{aligned}
A' &= A\cos^2\theta + \frac{B}{2}\sin(2\theta) + C\sin^2\theta \\
&= A\left(\frac{(A-C)+BH}{2BH}\right) + \frac{B^2}{2BH} + C\left(\frac{BH-(A-C)}{2BH}\right) \\
&= \frac{(A-C)^2+B^2+(A+C)BH}{2BH} \\
&= \frac{B^2H^2+(A+C)BH}{2BH} = \frac{A+C+BH}{2}.
\end{aligned}$$

Similarly,

$$C' = \frac{A+C-BH}{2}$$

and the full formula is as follows.

$$A' = \frac{A+C+BH}{2},\ B' = 0,\ C' = \frac{A+C-BH}{2}$$

$$D' = D\sqrt{\frac{BH+(A-C)}{2BH}} + E\sqrt{\frac{BH-(A-C)}{2BH}},$$

$$E' = E\sqrt{\frac{BH+(A-C)}{2BH}} - D\sqrt{\frac{BH-(A-C)}{2BH}},$$

$$F' = F$$

It follows that F and $A+C$ are invariant under rotation of axes by our angle θ. For $A'+C' = \dfrac{A+C+BH}{2} + \dfrac{A+C-BH}{2} = A+C$. Also, the discriminant B^2-4AC is invariant.

$$\begin{aligned} B'^2 - 4A'C' &= -4\left(\frac{A+C+BH}{2}\right)\left(\frac{A+C-BH}{2}\right) \\ &= [BH-(A+C)][BH+(A+C)] \\ &= B^2H^2-(A+C)^2 \\ &= B^2+(A-C)^2-(A+C)^2 \\ &= B^2-4AC \end{aligned}$$

If the graph is a hyperbola, then $-4A'C' = B^2-4AC > 0$ and if the graph is an ellipse $-4A'C' = B^2-4AC < 0$. And, if the graph is a parabola, then $-4A'C' = B^2-4AC = 0$.

Example 1 *Identify as a conic section and sketch the graph of* $x^2 - \sqrt{3}xy + 2y^2 - 2x = 0$. *We have* $A = 1$, $B = -\sqrt{3}, C = 2$, $D = -2$, *and* $E = F = 0$. *For* $\theta = \frac{1}{2}\cot^{-1}\left(\frac{A-C}{B}\right) = \frac{1}{2}\cot^{-1}\left(\frac{1}{\sqrt{3}}\right) = 30^o$ *we have* $BH = -\sqrt{3}\sqrt{\left(\frac{1-2}{-\sqrt{3}}\right)^2+1} = -\sqrt{3}\left(\frac{2}{\sqrt{3}}\right) = -2$ *and* $B^2-4AC = -5 < 0$ *so the graph cannot be either a parabola or a hyperbola, but may be an ellipse. Invoking the prime coefficients formula we have*

$$A' = \frac{1}{2},\ B' = 0,\ C' = \frac{5}{2},\ D' = -\sqrt{3},\ E' = 1,\ \text{and } F' = 0.$$

The equation relative to the rotated coordinate system is

$$\frac{1}{2}x'^2 + \frac{5}{2}y'^2 - \sqrt{3}x' + y' = 0.$$

Equivalently,

$$x'^2 + 5y'^2 - 2\sqrt{3}x' + 2y' = 0.$$

Completing the square we have $\left(x'-\sqrt{3}\right)^2 + 5\left(y'+\frac{1}{5}\right)^2 = \frac{16}{5}$, *or*

$$\frac{\left(x'-\sqrt{3}\right)^2}{\left(\frac{4}{\sqrt{5}}\right)^2} + \frac{\left(y'+\frac{1}{5}\right)^2}{\left(\frac{4}{5}\right)^2} = 1.$$

Indeed, the graph is a horizontal ellipse in the rotated system centered at $(x', y') = \left(\sqrt{3}, -\frac{1}{5}\right)$. *In the original coordinate system, the center is*

$$(x, y) = (\sqrt{3}\cos 30^o + \frac{1}{5}\sin 30^o, \sqrt{3}\sin 30^o - \frac{1}{5}\cos 30^o)$$

$$= \left(\frac{3}{2} + \frac{1}{10}, \frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{10}\right) = \left(\frac{8}{5}, \frac{2\sqrt{3}}{5}\right).$$

In this coordinate system the ellipse is not horizontal but cocked or tilted 30^o *counterclockwise. The* x*-intercepts are* $(0,0)$ *and* $(2,0)$. *The only* y*-intercept is* $(0,0)$. *The ellipse is tangent to the* y*-axis at the origin.*

Example 2 *Recognize the graph of* $y = \dfrac{x}{x-2}$ *as a hyperbola. We have equivalently, for* $x \neq 2$, $xy - x - 2y = 0$ *so that* $A = C = 0$, $B = 1$, $D = -1$, $E = -2$, *and* $F = 0$. *If* $\theta = \frac{1}{2}\cot^{-1}(0) = 45^o$ *is the angle of rotation of the* x, y*-coordinate system, we have* $BH = 1$ *so that*

$$A' = \frac{1}{2},\ B' = 0,\ C' = -\frac{1}{2},\ D' = -\frac{3}{\sqrt{2}},\ E' = -\frac{1}{\sqrt{2}},\ \text{and } F' = 0.$$

This yields the rotated equation $\frac{1}{2}x'^2 - \frac{1}{2}y'^2 - \frac{3}{\sqrt{2}}x' - \frac{1}{\sqrt{2}}y' = 0$ *or*

$$x'^2 - y'^2 - 3\sqrt{2}x' - \sqrt{2}y' = 0.$$

Completing the square gives us

$$\left(x' - \frac{3}{\sqrt{2}}\right)^2 - \left(y' + \frac{1}{\sqrt{2}}\right)^2 = \frac{9}{2} - \frac{1}{2} = 4$$

so that indeed the graph is a hyperbola with rotated equation

$$\frac{\left(x' - \frac{3}{\sqrt{2}}\right)^2}{2^2} - \frac{\left(y' + \frac{1}{\sqrt{2}}\right)^2}{2^2} = 1.$$

The center is at $(x', y') = \left(\frac{3}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$ *in the rotated system which in the unrotated system is*

$$(x, y) = \left(\frac{3}{\sqrt{2}}\cos 45^o + \frac{1}{\sqrt{2}}\sin 45^o, \frac{3}{\sqrt{2}}\sin 45^o - \frac{1}{\sqrt{2}}\cos 45^o\right)$$

$$= \left(\frac{3}{2} + \frac{1}{2}, \frac{3}{2} - \frac{1}{2}\right) = (2, 1).$$

The asymptotes in the rotated system are the lines

$$y' + \frac{1}{\sqrt{2}} = \pm\left(x' - \frac{3}{\sqrt{2}}\right)$$

which leads to

$$y' = x' - 2\sqrt{2} \text{ or } y' = -x' + \sqrt{2}.$$

Now, by rotating the x', y'*-system through the angle* $-\theta$ *we have*

$$x' = x\cos(-\theta) - y\sin(-\theta) \text{ and } y' = x\sin(-\theta) + y\cos(-\theta)$$

or

$$x' = x\cos\theta + y\sin\theta, \text{ and}$$
$$y' = -x\sin\theta + y\cos\theta.$$

Thus, the equations of the asymptotes in the x, y*-system are* $-x\sin 45^o + y\cos 45^o = x\cos 45^o + y\sin 45^o - 2\sqrt{2}$ *or* $-x\sin 45^o + y\cos 45^o = -x\cos 45^o - y\sin 45^o + \sqrt{2}$. *These equations simplify to*

$$\frac{\sqrt{2}}{2}(-x+y) = \frac{\sqrt{2}}{2}(x+y-4) \text{ or } \frac{\sqrt{2}}{2}(-x+y) = \frac{\sqrt{2}}{2}(-x-y+2)$$

or

$$x = 2 \text{ or } y = 1$$

which agrees with the known asymptotes for this rational function. Similarly, we can find the two lines of symmetry for the graph of our rational function in the x, y*-system. In the* x', y'*-system, the lines of symmetry are* $x' = \frac{3}{\sqrt{2}}$ *and* $y' = -\frac{1}{\sqrt{2}}$. *In the* x, y*-system these are the equations*

$$x\cos 45^o + y\sin 45^o = \frac{3}{\sqrt{2}} \text{ and } -x\sin 45^o + y\cos 45^o = -\frac{1}{\sqrt{2}}$$

which simplify to

$$x + y = 3 \text{ and } x - y = 1.$$

We now give the code for a program to be executed on the TI-83 or TI-84 graphing calculator.

```
: ClrHome
: Degree
: Input  "A COEF = ", I
: ClrHome
: Input "B COEF = ", J
: ClrHome
```

```
: Input "C COEF = ", K
: ClrHome
: Input "D COEF = ", L
: ClrHome
: Input "E COEF = ", M
: ClrHome
: Input "F COEF = ", N
: ClrHome
: If J = 0
: Then
: Disp  {I, J, K, L, M, N}
: Else
```

: $\sqrt{(((I-K)\ /\ J)^2+1)} \rightarrow H$

: $(J*H+I+K)\ /\ 2 \rightarrow A$

: $0 \rightarrow B$

: $(I+K-J*H\ /\ 2 \rightarrow C$

: $L*\sqrt{((J*H+I-K)\ /\ (2*J*H))}+M*\sqrt{((J*H-(I-K))\ /\ (2*J*H))} \rightarrow D$

: $M*\sqrt{((J*H+I-K)\ /\ (2*J*H)}-L*\sqrt{((J*H-(I-K))\ /\ (2*J*H)} \rightarrow E$

: $N \rightarrow F$

: $(\cos^{-1}((I-K)\ /\ (J*H))\ /\ 2 \rightarrow T$

```
: Output (1, 1, "A = ")
: Output (1, 5, A)
: Output (2, 1, "B = ")
: Output (2, 5, B)
: Output (3, 1, "C = ")
: Output (3, 5, C)
: Output (4, 1, "D = ")
: Output (4, 5, D)
: Output (5, 1, "E = ")
: Output (5, 5, E)
: Output (6, 1, "F = ")
: Output (6, 5, F)
: Output (7, 1, "ANGLE = ")
: Output (7, 7, T)
: Output (8, 1, "Discr = ")
: Output (8, 7, −4 * A * C )
: {A, B, C, D, E, F}→L1
: L1² → L2
: Radian
```