Want an Active Classroom?
Use the Five Practices!

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Basic idea

• Students are given some form of preparation before class.
• In class, provide cognitively demanding tasks.
  • Students work in groups (sometimes pairs)
  • Work is usually on white boards
• Students share their solutions with the class.*
• Use student solutions/approaches to meet the goals of the lesson.
Five practices (Smith & Stein, 2011)

**Practice 0:** Set a Goal

**Practice 1:** Anticipation

**Practice 2:** Monitoring

**Practice 3:** Selecting

**Practice 4:** Sequencing

**Practice 5:** Connecting

Legend

- **Before Class**
- **Planning to Teach/Students are Engaged (In Class)**
- **Teaching is Visible (In Class)**
Takeaway

• Practice “n” will run smoothly as long as Practice “n – 1” is done well.

• Tasks need to be high level.

• Students’ mistakes are important (i.e., making them and learning from them).

• 5 Practices ≠ Show and Tell
First Day: Establishing the Culture

Give them a task that...
- Immediately gets them talking/discussing
- Fosters productive struggle
- Is amenable to many different strategies
Patterns (Any class)

(a) Describe how you see the shapes growing.
(b) How many squares will you see in the fifth diagram?
(c) How many squares will you see in the nth diagram?
(d) Will there ever be a diagram with exactly 200 squares? If not, which diagram contains the closest to 200 squares?

Source: Youcubed
CAN YOU FIND $A$? (Trigonometry/Precalculus)

Look carefully at this equation:

$$[3(230 + A)]^2 = 49280A$$

What is the value of the digit $A$ and HOW did you arrive at your answer?

Source: nrich.maths.org
Football, anyone? (Calculus)

Take a long rope, tie it to the bottom of the goal post at one end of a football field. Then run it across the length of the field (120 yards) to the goal post at the other end. Stretch it tight, and then tie it to the bottom of that goal post, so that it lies flat against the ground.

Now suppose you add one foot of slack to the rope, so that now you can lift it off the ground at the 50-yard line. How high can you lift it up?

(a) Not high enough to fit my finger under it.
(b) Just high enough to crawl under.
(c) Just high enough to walk under.
(d) High enough to drive a truck under.

JUSTIFY YOUR ANSWER!!

Source: Math Fun Facts, Francis Su
How Far? (Calculus)

A car travels downhill at 72 mph, on level ground at 63 mph, and uphill at 56 mph. The car takes 4 hours to travel from town A to town B. The return trip takes 4 hours and 40 minutes.

Find the distance between the two towns.
Problem Solving (Calculus)

The cube below has a diagonal $d$ of length 13.85 cm. What is the length of one of the sides of the cube?

Source: Ollie Lovell
• Growth mindset
  “I can do this.”
• Everyone can succeed
• Understanding
• Making mistakes
• Working hard
• Authentic learning

• Fixed mindset
  “I’m bad at word problems.”
• The “math gene”
• Memorization
• Illusion of fluency
• Being smart
• Quick/artificial categorization of a problem
Think about....

• **Being part of a group means...**
  • Accepting being “stuck” as a normal part of learning
  • Providing input to the group: What’s your contribution?
  • *** Sharing with the class when you have something good! ***

• **Difficult tasks...**
  • ...often require input from many people
  • ...may take longer to solve

• Deep, slow thinking > Shallow, quick thinking
What Do the Five Practices Look Like?
All Shook Up

Nine people meet at a party. They all exchange handshakes. How many handshakes are exchanged?

Source: Shutterstock
Section 1: Jan 23:

9 people meet at a party and exchange handshakes.
How many handshakes occurred?

36 handshakes
36

\[ \frac{36}{8} = 4.5 \]

\[ \frac{45}{9} = 5 \]

\[ \frac{56}{8} = 7 \]

\[ \frac{67}{8} = 8.375 \]

\[ \frac{78}{9} = 8.666\ldots \]

\[ \frac{89}{10} = 8.9 \]

\[ \frac{90}{10} = 9 \]

\[ \frac{100}{10} = 10 \]
8 7 6 5 4 3 2 1 0 = 36
Cutting & Stacking Paper

Take a sheet of paper, cut it in half (maybe fold it first), and then place one half on top of the other half to create a pile of paper with a height equal to the thickness of two sheets of paper. Take that pile, cut it in half (maybe fold it first), and place one half on top of the other. The resulting pile would have a height equal to four sheets of paper. Continue this process...

Question: How high would the pile be if you repeated this 50 times?

Source: Trinter & Garofalo, 2011
<table>
<thead>
<tr>
<th># of folds</th>
<th># of layers</th>
<th>Formula</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>(2^{0} = 1)</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
<td>(2^{1} = 2)</td>
</tr>
<tr>
<td>2</td>
<td>4</td>
<td>(2^{2} = 4)</td>
</tr>
<tr>
<td>3</td>
<td>8</td>
<td>(2^{3} = 8)</td>
</tr>
<tr>
<td>4</td>
<td>16</td>
<td>(2^{4} = 16)</td>
</tr>
<tr>
<td>5</td>
<td>32</td>
<td>(2^{5} = 32)</td>
</tr>
<tr>
<td>6</td>
<td>64</td>
<td>(2^{6} = 64)</td>
</tr>
</tbody>
</table>

# = # of folds
P = layers of paper
T = total thickness

\(2^{6} = 1.1258999 \times 10^{6}\)

\(4.503996 \times 10^{2}\)
<table>
<thead>
<tr>
<th>Folds</th>
<th>Layers</th>
<th>Thickness</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>.004</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
<td>.008</td>
</tr>
<tr>
<td>2</td>
<td>4</td>
<td>.016</td>
</tr>
<tr>
<td>3</td>
<td>8</td>
<td>.032</td>
</tr>
<tr>
<td>4</td>
<td>14</td>
<td>.064</td>
</tr>
</tbody>
</table>

\[2^F = L\]

\[L \cdot 0.004 = T\]

\[T = 2^F (0.004)\]

\[T = 4\text{ trillion inches}\]

\[\frac{T}{12} = 375\text{ billion ft}\]

\[\frac{T}{5280} = T\text{ in Mi}\]

71,079,539.5733377 Miles
Problem

TRUE / FALSE

Given $f(x) = \frac{x^2 - 4}{x - 2}$ and $g(x) = x + 2$, we can say the functions $f$ and $g$ are equal. Explain your reasoning!
The graphs won't be equal but they will be pretty close to looking the same. There needs to be restrictions on the good graph for them to be equal.

FALSE

\[
f(x) = \frac{x^3 - 4}{x - 2} = \frac{x + 2}{x/2}
\]

\[
f'(x) = 2x + 2, \quad x \neq 2
\]

false cause they are equal, but not at \( x = 2 \)
\[ \frac{x^2 - 4}{x - 2} = x + 2 \]

Undefined \[ @ x = 2 \]

False \[ @ x = 2 \]

4
Problem

TRUE / FALSE

If \( f(x) \leq g(x) \) for all \( x \neq a \),
then \( \lim_{x \to a} f(x) \leq \lim_{x \to a} g(x) \).

Justify!!
True,

\[ g(x) \]
\[ f(x) \]
\[ a \]

\[ FALSE - b/c \]
\[ \lim_{x \to a} f(x) = y \]
\[ \lim_{x \to a} g(x) = y \]
\[\lim_{x \to a} f(x) \leq \lim_{x \to a} g(x)\]

False

\[\lim_{x \to a} g(x) > \lim_{x \to a} f(x)\]

true option

\[\lim_{x \to a} g(x) = \lim_{x \to a} f(x)\]

false option
Problem

True/False/Discuss
At some time since you were born, your weight in pounds equaled your height in inches.
True
You can't bypass 3 ft.

True, because at one point we were shorter than 3 feet and now we are taller than 3 feet so we must have been exactly 3 feet at one point.
If the graph below is the graph of \( y = f(x) \), which of the following (a, b, c, or d) is most likely the graph of \( y = f'(x) \)?
overall upward trend

a)
goes from neg to pos but not via 0

slope is negative at first so the derivative will be negative until the bottom point on the graph, so the derivative switches from negative to positive at the bottom point on the x-axis.
Problem

You already know that
\[
\frac{d}{dx} \left( \frac{f(x)}{g(x)} \right) = \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}.
\]

Expressing \( \frac{f(x)}{g(x)} = f(x) \cdot [g(x)]^{-1} \), differentiate this latter form to show agreement with the Quotient Rule.
\[
\frac{f(x)}{g(x)} = f(x) \cdot \left[ g(x) \right]^{-1} \cdot g'(x)
\]

\[
\frac{d}{dx} \left[ \frac{f(x)}{g(x)} \right] = f(x) \cdot \left[ g(x) \right]^{-2} + \left[ g(x) \right]^{-1} \cdot f'(x)
\]

\[
'' = f(x) \cdot \frac{-1}{g(x)^2} + \frac{f(x)}{g(x)}
\]

\[
'' = \frac{-f(x)}{\left[ g(x)^2 \right]^2} + \frac{f'(x)}{g(x)} \cdot \frac{g(x)}{g(x)}
\]

\[
'' = \frac{f'(x)g(x) - f(x)g'(x)}{\left[ g(x)^2 \right]^2} = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}
\]
\[
\frac{d}{dx} \left( f(x)g(x) \right)^{-1} = \\
(f(x))^{-1}(-g(x))^{-2}g'(x) + (g(x))^{-1}(f'(x)) \\
- \frac{f(x)g'(x)}{(g(x))^2} + \frac{f'(x)g(x)}{g(x)g(x)} \\
\]

Common denom.

\[
f'(x)g(x) - f(x)g'(x)
\]

\[
(\frac{1}{(g(x))^2})
\]
Problem

A machine is causing a particle to move along the $x$-axis so that its position at time $t$ is given by $x(t) = (t - 4)^2$, where $t$ is in seconds.

(a) What is the particle's velocity at $t = 2$? Interpret.

(b) The machine stops suddenly at $t = 3$, releasing the particle. As the particle continues, where will it be 5 seconds after the machine stops? Explain your thinking.
a. \( x'(t) = 2t + 8 \)
\[ x'(2) = 2(2) - 8 = -4 \]

b. \( x(0) = 1 \)
\[ x'(3) = -2 \text{ m/s} \]

After 5 sec of the machine off, the particle would be at \( x = -9 \). The position of the particle at 3 seconds would be 1 unit. The particle’s velocity is -2 m/s at 3 seconds. The particle continues at -2 m/s for 5 sec which means it moves -10 units. -10 units + 1 unit, the starting position, will put you at -9 units.

\[ x(t) = (t - 4)^2 \]
\[ v(t) = 2(t - 4)(1) = 2t - 8 \]
\[ v(2) = -4 \text{ m/s} \]

\[ a(t) = 2 \text{ m/s}^2 \] (while machine is working)

\[ v(3) = -2 \text{ m/s} \] (right when the machine stops)

After machine stops, EF becomes zero, so acceleration becomes zero, so it remains at that velocity it had (-2 m/s).

\[ \Delta x \text{ in first } 3\text{s:} (3-4)^2 = 1\text{m} = x_0 \]

\[ x \text{ afterwards: } x = vt + x_0 = -2(5) + 1 = -9 \text{ m} \]
\[ x(t) = (t-4)^2 \]
\[ v = 2(t-4) \]
\[ v = 2t - 8 \]
\[ v = 2(3) - 8 \]
\[ v = -4 \]

\[ \begin{align*}
v &= 2t - 8 \\
v &= 2(3) - 8 \\
v &= -2 = m \\
1 &= -2(3) + b \\
1 &= -6 + b \\
7 &= b \\
v &= -20t + 7 \rightarrow \theta \ 8 \sec (3 + 5) \\
v &= -2(8) + 7 \\
v &= -16 + 7 \\
v &= -9 \end{align*} \]
Warm Up

Sketch the graph of a function \( y = f(x) \) that has each of the given characteristics below.

(a) \( f(2) = f(4) = 0 \)
(b) \( f'(x) < 0 \) if \( x < 3 \)
(c) \( f'(3) \) does not exist
(d) \( f'(x) > 0 \) if \( x > 3 \)
(e) \( f''(x) < 0 \) for \( x \neq 3 \)
a) \( f(2) = 0 \)
\( f(4) = 0 \)

b) \( x < 3 \) decreasing

c) \( f'(3) \) DNE

d) \( x > 3 \) increasing

e) \( x \neq 3 \) concave up
a) Zeros @ x = 2 and 4
b) dec. left of 3
c) not differentiable @ x = 3
d) inc. right of 3
E) concave down where x ≠ 3

or

[Diagram of a function with arrows and points labeled 2, 3, 4]
Problem

Evaluate \( \lim_{x \to \infty} \frac{\sin x}{x} \).

Compare/contrast this with the problem \( \lim_{x \to \infty} \sin x \).
\[ \lim_{x \to \infty} \frac{\sin x}{x} = 0 \]

As \( x \) gets larger, the function gets closer to zero. As \( x \) gets larger, the function doesn't get close to any function, it keeps a wave motion.

\[ \lim_{x \to \infty} = 0 \]

Because the values of \( \sin x \) by itself follow a constant trend, dividing it by \( x \) causes the amplitude of our function to shrink as \( x \) grows. As it continues to \( \infty \) the values come arbitrarily close to 0.

\[ \lim_{x \to \infty} \frac{\sin x}{x} = 0 \]
\[\lim_{x \to \infty} \sin x = \text{DNE}\], sine is a periodic function that does not have any asymptotes.

\[\lim_{x \to \infty} \frac{\sin x}{x} = 0\]  
\[\lim_{x \to \infty} \frac{\sin x}{x} = -\frac{1}{\infty} = 0\]

\[\lim_{x \to \infty} \frac{\sin x}{x} = \frac{\sin x}{x} \quad \text{will always be between 1 and -1, so as } x \to \infty \text{ there will be two limits}\]

\[\frac{1}{\infty} \text{ and } -\frac{1}{\infty}\].

This means that normally the limit would not exist, but using \(L(x) = \frac{1}{x}\) and \(h(x) = \frac{-1}{x}\) the squeeze theorem can force

\[g(x) = \frac{\sin x}{x}\] to approach a limit of zero.

\[h(x) = \frac{1}{x}\] \[L(x) = \frac{1}{x}\] \[\lim_{x \to \infty} L(x) = 0\] and \[\lim_{x \to \infty} h(x) = 0\] therefore \[\lim_{x \to 0^+} g(x) = 0\]

\[\lim_{x \to 0^-} g(x) = 0\]
Problem

(a) Oil leaks from a tank at the rate of \( r(t) \) gallons per minute at time \( t \).

What is the meaning of \( \int_{0}^{120} r(t) \, dt \)? What are its units?

(b) If an object moves along a straight path with velocity \( v(t) \) meters/sec,

what is the meaning of \( \int_{2}^{5} v(t) \, dt \)? What are its units?
A) \( \int_{0}^{120} r(t) \, dt \) explains how much oil has leaked out (gallons).

B) \( \int_{2}^{5} v(t) \, dt \) explains the distance traveled in the time interval (meters).

a) Total volume leaked out in 2 hrs gallons
b) Distance traveled between 2 and 5 seconds meters.
Antiderivatives

If \( f \) is an antiderivative of \( g \), and \( g \) is an antiderivative of \( h \), then
(a) \( h \) is an antiderivative of \( f \).
(b) \( h \) is the second derivative of \( f \).
(c) \( h \) is the derivative of \( f'' \).

Source: Cornell’s Good Questions
\[ f(x^2 + 2x + 1) \]

\[ f \xrightarrow{g} h \]

\[ f = g \quad \iff \quad f'' = h \quad \Rightarrow \quad f'' = h \quad \Rightarrow \quad f'' = f'' \]
Example

Consider the "Top Curve – Bottom Curve" formula for the picture. Does this formula apply in this situation? Why/why not?
No

could be \(-1^2 \neq f\)

No

Steps at x axis needs to be added to work
(+) value

(-) value

\[ +v - (-v) = +v \]

yes

Total area between curves

The formula does apply. There would be a positive area minus a negative one. This would yield to positive areas being added together.

\[
\int_{a}^{b} (f(x) - g(x)) \, dx = \int_{a}^{b} (f(x) + g(x)) \, dx
\]
Yes you can. You can use \( y = 0 \) as another curve. This can allow you to use \( y = 0 \) as the bottom with the equation \( f(x) \) and as the top curve for the equation \( g(x) \).

\[
\text{Total Area} = \int_a^b f(x) - (y = 0) - \int_a^b (y = 0) - g(x)
\]
Calculus II
Problem

Carefully sketch the graph of \( y = 2^x \) (e.g., plot at least 4 points). Use this information to also sketch \( y = \log_2 x \) on the same set of axes.
\[ y = 2^x \]

\[ y = \log_2 x \]

<table>
<thead>
<tr>
<th>( x )</th>
<th>1</th>
<th>2</th>
<th>4</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y )</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
</tbody>
</table>
$g(x) = \log_2 x$
$D = \mathbb{R} \times (0, \infty)$
$R = \mathbb{R} \times (0, \infty)$

$f(x) = 2^x$
$D = (-\infty, \infty) \times \mathbb{R}$
$R = \mathbb{R} \times (0, \infty)$
Problem

Solve the following equations by using the techniques/properties of logs and exponentials:

(a) \( e^{0.08t} = 2500 \)

(b) \( \log_3 (2x - 1) - \log_3 (x - 4) = 2 \)

(c) \( e^x + e^{-x} - 6 = 0 \)
\[ e^{0.08t} = 2500 \]
\[
\ln e^{0.08t} = 2500 \\
0.08t = \ln 2500 \\
t = \frac{\ln 2500}{0.08} \quad t = 97.8
\]

\[ \log_3 (2x-1) - \log_3 (x-4) = 2 \]
\[ \log_3 \left( \frac{2x-1}{x-4} \right) = 2 \]
\[ 3^2 = \frac{2x-1}{x-4} \quad 9(x-4) = 2x-1 \]
\[ 9x - 36 = 2x - 1 \quad 7x = 35 \]
\[ x = 5 \]
C. \( e^x + e^{-x} - 6 = 0 \)

\[ y = e^x \]
\[ e^x + \frac{1}{e^x} = 6 \]
\[ y + \frac{1}{y} = 6 \]
\[ \frac{y^2 + 1}{y} = 6 \]
\[ y^2 + 1 = 6y \]
\[ y^2 - 6y + 1 = 0 \]
\[ x = 1.762 \quad x = -1.762 \]

\[ y = \frac{6 \pm \sqrt{36 - 4}}{2} \]
\[ y = \frac{6 \pm \sqrt{32}}{2} \]
\[ y = 5.828 \quad y = 1.171 \]

\[ e^x = 5.828 \quad e^x = 1.171 \]
\[ x \approx \pm 1.7627 \]
\[ x = \ln \left( 3 \pm 2\sqrt{2} \right) \]
\[ \cosh(?) = 3 \]
Problem

(a) \[ \int e^x \sin x \, dx \]

(b) \[ \int x^2 \sin x \, dx \]
\[
\int e^x \sin x \, dx \quad u = e^x \quad dv = \sin x \, dx \\
\frac{du}{du} = e^x \quad v = -\cos x \\
\]

\[
= -e^x \cos x + \int \frac{e^x \cos x}{u} \, du \\
= -e^x \cos x + \left(e^x \sin x - \int e^x \sin x \, dx\right)
\]
Problems

(a) \( \int \cos^3 x \sin x \, dx \)  
(b) \( \int \cos^3 x \, dx \)

(c) \( \int \sin^5 t \cos^2 t \, dt \)  
(d) \( \int \tan^6 t \sec^4 t \, dt \)
a) \[ \int \cos^3 x \sin x \, dx \]

\[ -\int u^3 \, du \quad \text{with} \quad u = \cos x \]
\[ du = -\sin x \, dx \]
\[ -du = \sin x \, dx \]

\[ -\frac{u^4}{4} + C \]

\[ -\frac{1}{4} \cos^4 x + C \]
\[ \int \cos^3 x \sin x \, dx = -\frac{1}{4} \sin^4 x + \frac{1}{2} \sin^2 x + C \]

\[ \int \cos^2 x \cos x \sin x \, dx \]

\[ u = \sin x \]
\[ du = \cos x \, dx \]
\[ \int (1-u^2) \, du \]
\[ \int u^3 + u \, du \]
\[ -\frac{u^4}{4} + \frac{u^2}{2} + C \]

\[ \int (1-u^2) 
\]
\[ = \int \frac{\sin^2 x}{4} \, dx \]
\[ = \int \frac{\sin^4 x}{4} + C \]
Problem

Evaluate \( \int \frac{dx}{\sqrt{9 + x^2}} \).

Notice there is no “basic rule” that fits this integral!
\[
\int \frac{dx}{\sqrt{9 + x^2}} = \frac{x}{3} \tan^{-1} \left( \frac{x}{3} \right) + c
\]

\[
x = 3 \tan \theta
\]

\[
dx = 3 \sec^2 \theta \, d\theta
\]

\[
\sec \theta = \frac{1}{\cos \theta}
\]

\[
\cos \theta = \frac{a}{3}
\]

\[
\sec A = \frac{a}{q} = \frac{\sqrt{9 + x^2}}{3}
\]

\[
\sqrt{9 + x^2} = 3 \sec \theta
\]

\[
\int \frac{1}{3 \sec \theta} \cdot 3 \sec^2 \theta \, d\theta = \int \sec \theta \, d\theta
\]

\[
= \ln |\sec \theta \cdot \tan \theta| + c
\]
\[ S \frac{dx}{\sqrt{9 + x^2}} = 3 \tan \theta \]

\[ dx = 3 \sec^2 \theta d\theta \]

\[ \tan \theta = \frac{x}{3} \]

\[ \sqrt{9 + x^2} = 3 \sec \theta \]

\[ \int \frac{3 \sec^2 \theta}{3 \sec \theta} \, d\theta \]

\[ \sec \theta = \frac{\sqrt{9 + x^2}}{3} \]

\[ \int \sec \theta \, d\theta \]

\[ \ln | \sec \theta + \tan \theta | + C \]

\[ \ln \left| \frac{\sqrt{9 + x^2}}{3} + \frac{x}{3} \right| + C \]
Evaluate \( \int \frac{x-1}{x^2-x-2} \, dx \). First, find the partial fraction decomposition. Second, integrate.
\[ \int \frac{x-1}{(x-2)(x+1)} \, dx = \int \frac{A}{x-2} \, dx + \int \frac{B}{x+1} \, dx \]

\[ \begin{align*}
X-1 &= (x+1)A + (x-2)B \\
X-1 &= 1 \quad \text{for } x=2 \\
(A+2) &= \frac{3}{2} \\
A &= \frac{1}{3}
\end{align*} \]

\[ \int \frac{1}{x-2} \, dx + \int \frac{2}{x+1} \, dx = \ln|x-2| + \frac{2}{3} \ln|x+1| + C \]

\[ \sum_{i=0}^{n} \frac{x-1}{(x+1)(x-2)} = \int \frac{x-1}{(x+1)(x-2)} \, dx \]

\[ \begin{align*}
X-1 &= A(x-2) + B(x+1) \\
X-1 &= A(x-2) - 2A + Bx + B \\
2A + B &= -1 \\
A + B &= 1 \\
2A + 2B &= -2 \\
3B &= 1 \\
B &= \frac{1}{3} \\
A &= \frac{2}{3}
\end{align*} \]

\[ \begin{align*}
\sum_{i=0}^{n} \frac{x-1}{(x+1)(x-2)} &= \int \frac{x-1}{(x+1)(x-2)} \, dx \\
\frac{1}{3} \sum_{i=0}^{n} (x+1) + \frac{1}{3} \sum_{i=0}^{n} (x-2) &= \frac{2}{3} \int_{x=1}^{x=n} (x+1) + \frac{1}{3} \int_{x=1}^{x=n} (x-2) + C
\end{align*} \]
\[ \int \frac{x-1}{x^2-x-2} \, dx = \int \frac{x-1}{(x-2)(x+1)} \, dx \]

\[ \frac{x-1}{(x-2)(x+1)} = \frac{A}{x-2} + \frac{B}{x+1} \]

\[ \therefore x-1 = A(x+1) + B(x-2) \]

\[ x-1 = (A+B)x + A-2B \]

\[ \therefore A+B = 1 \]

\[ \frac{A-2B = 1}{3B = 2} \Rightarrow \begin{cases} A = \frac{2}{3} \\ B = \frac{1}{3} \end{cases} \]

\[ \therefore \int \frac{x-1}{(x-2)(x+1)} \, dx = \int \left( \frac{\frac{2}{3}}{x-2} + \frac{\frac{1}{3}}{x+1} \right) \, dx \]

\[ = \frac{1}{3} \int \left( \frac{1}{x-2} + \frac{1}{x+1} \right) \, dx \]

\[ = \frac{1}{3} \left[ \ln|x-2| + 2\ln|x+1| \right] + C \]
Problem

Find the form for the partial fraction decomposition.

Example: Given \( \frac{x^2 + 5x + 2}{(x+1)(x^2+1)} \), we can write \( \frac{A}{x+1} + \frac{Bx+C}{x^2+1} \)

(a) \( \frac{x+2}{x(x+1)^2} \)

(b) \( \frac{x^2 + 4x + 1}{x^3(x+1)(x^2+4)} \)

(c) \( \frac{x^2 - 3x + 7}{(x^2 + 9)^2} \)
(a) \[ \frac{x+2}{x(x+1)^2} = \frac{A}{x} + \frac{B}{(x+1)} + \frac{B_2}{(x+1)^2} \]

(b) \[ \frac{x^3+4x+1}{x^3(x+1)(x^2+4)} = \frac{A}{x^3} + \frac{B}{(x+1)} + \frac{C}{(x^2+4)} \]

(c) \[ \frac{x^2-3x+7}{(x^2+9)(x^2+9)} = \frac{A}{(x^2+9)} + \frac{Bx+C}{(x^2+9)} + \frac{Ax+c}{(x^2+9)^2} + \frac{4}{(x^2+9)^2} \]
a) \[ \frac{x+2}{x(x+1)^2} \Rightarrow \frac{A}{x} + \frac{B_1}{x+1} + \frac{B_2}{(x+1)^2} \]

b) \[ \frac{x^2 + 4x + 1}{x^3(x+1)(x^2+4)} \Rightarrow \frac{A}{x+1} + \frac{B_1}{x} + \frac{B_2}{x^2} + \frac{B_3}{x^3} + \frac{C_1x + D}{x^2 + 4} \]

c) \[ \frac{x^2 - 3x + 7}{(x^2 + 9)^2} \Rightarrow \frac{B_1x + C_1}{x^2 + 9} + \frac{B_2x + C_2}{(x + 9)^2} \]
Problem

Produce an example of each below. A sequence that is...

(a) both bounded and monotonic.
(b) bounded but not monotonic.
(c) not bounded but monotonic.
(d) neither bounded nor monotonic.

Miscellaneous thought: We know a bounded and monotonic sequence converges. However, must a convergent sequence be both bounded and monotonic?
a. \( \left\{ \frac{1}{n} \right\}_{n=1}^{\infty} \)

b. \( \left\{ (-1)^n \right\}_{n=1}^{\infty} \)

c. \( \left\{ n \right\}_{n=1}^{\infty} \)

d. \( \left\{ n \cos(n) \right\}_{n=1}^{\infty} \)

\[ \sum \frac{1}{n^3}, \frac{1}{8}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5} \]

\[ \{ \cos \pi n \} \]

\(-1, 1, -1, 1, \ldots \)

d. \( \left\{ (-2)^n \right\} \)...

\(2, 4, 8, 16\)

\(-\infty, 2, 4, 8, 16, \ldots\)

\(-\infty < (2)^n < 0\)
Problem

One of these series definitely diverges; the other is questionable. Explain which is which!!

\[ \sum_{n=1}^{\infty} \frac{n}{n^2 + 1} \]  and  \[ \sum_{n=1}^{\infty} \frac{n^2}{n^2 + 1} \]
\[
\lim_{n \to \infty} \frac{n^2}{n^2 + 1} = \frac{\infty}{\infty} = 2n - \frac{2n}{2n} = 1 \neq 0
\]

The series diverges because the limit does not equal 1.

\[
\lim_{n \to \infty} \frac{n}{n^2 + 1} = \frac{\infty}{\infty} = \frac{1}{2n} \to 0
\]

We cannot be certain that this series diverges because the limit equals 0.

\[
\sum_{n=1}^{\infty} \frac{n}{n^3 + 1} = \frac{\infty}{\infty} = 0
\]

\[
\lim_{n \to \infty} \frac{n^2}{n^2 + 1} = \frac{\infty}{\infty} = \frac{8}{8} = 1
\]

\[
\lim_{n \to \infty} \frac{2n}{2n} = \frac{1}{1} = 1
\]

\[
\lim_{n \to \infty} \frac{n}{n^2 + 1} = 0
\]

\[
\lim_{n \to \infty} \frac{n}{n^3 + 1} = 0
\]

\[
\sum_{n=1}^{\infty} \frac{n^2}{n^3 + 1} = 170
\]

Diverges
\[ \sum_{n=1}^{\infty} \frac{n}{n^2+1} \left( \frac{1}{4+1} + \frac{2}{9+1} + \frac{3}{16+1} + \frac{4}{25+1} + \cdots \right) \]

\[ \frac{1}{5} + \frac{2}{5} + \frac{3}{10} + \frac{4}{17} \quad \text{going to 0} \]

\[ \lim_{n \to \infty} = 0 \quad \text{Convergent} \]

\[ \sum_{n=1}^{\infty} \frac{n^2}{n^3+1} \left( \frac{1}{4+1} + \frac{4}{9+1} + \frac{9}{16+1} + \frac{16}{25+1} + \cdots \right) \]

\[ \frac{1}{5} + \frac{4}{5} + \frac{9}{10} + \cdots \quad \text{going to 0} \]

\[ \lim_{n \to \infty} = 1 \quad \text{Convergent} \]

\[ \lim_{n \to \infty} = 1 \quad \text{Divergent} \]
Problem

Look at the following $p$-series. Do they converge or diverge? Use Desmos/graphing calculator to support this by showing partial sums $s_{20}$, $s_{50}$, and $s_{125}$.

(a) $\sum_{n=1}^{\infty} \frac{1}{n^2}$  \hspace{1cm} (b) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$  \hspace{1cm} (c) $\sum_{n=1}^{\infty} \frac{1}{n}$
Converge \( p > 1 \)
Diverge \( p \leq 1 \)

a) \( \sum \frac{1}{n^2} \rightarrow \text{Converge} \)
\[
\begin{align*}
S_{20} &= 1.596 \\
S_{50} &= 1.625 \\
S_{125} &= 1.637
\end{align*}
\]
\[\approx 1.64\]

b) \( \sum \frac{1}{n^{0.5}} \rightarrow \text{Diverge} \)
\[
\begin{align*}
S_{20} &= 7.595 \\
S_{50} &= 12.752 \\
S_{125} &= 20.945
\end{align*}
\]

c) \( \sum \frac{1}{n} \rightarrow \text{Diverge} \)
\[
\begin{align*}
S_{20} &= 3.697 \\
S_{50} &= 4.499 \\
S_{125} &= 5.410
\end{align*}
\]

\[\frac{1}{n^2} \quad p = 2 > 1 \quad \text{Converges by P-Series Test} \]
\[
\frac{1}{n} \quad p = \frac{1}{2} \leq 1 \quad \text{Diverges by P-Series Test} \]

\[
\begin{align*}
\sum_{n=1}^{20} \frac{1}{n^2} &= 1.5962 \\
\sum_{n=1}^{50} \frac{1}{n^2} &= 1.6251 \\
\sum_{n=1}^{125} \frac{1}{n^2} &= 1.6370
\end{align*}
\]

\[
\begin{align*}
\sum_{n=1}^{20} \frac{1}{n} &= 7.5953 \\
\sum_{n=1}^{50} \frac{1}{n} &= 12.7524 \\
\sum_{n=1}^{125} \frac{1}{n} &= 20.9450
\end{align*}
\]

\[
\begin{align*}
\sum_{n=1}^{20} \frac{1}{n} &= 3.5977 \\
\sum_{n=1}^{50} \frac{1}{n} &= 4.4992 \\
\sum_{n=1}^{125} \frac{1}{n} &= 5.4083
\end{align*}
\]
Sample feedback (Calculus I)

- Classroom facilitated learning in a hands-on manner. Allowed students to test their knowledge as well as inspired critical thinking.

- I like how the professor put the class into groups to try and solve problems together with peers instead of constant presentation style instruction. I also like how the professor would be available to talk or discuss issues or questions outside of class.

- I liked how we used the whiteboards frequently during class. It gave me an opportunity to work on problems during class, which really helped me understand the concepts that were being taught in class.

- I liked that we worked in groups all the time. We were constantly working on problems instead of being lectured which I think is a good thing to keep people focused, plus the class periods went by super fast.

- I very much liked using whiteboards in class.

- Interactive activities during class, i.e sharing thoughts and ideas towards solving Mathematical problems.

- It was a different style of teaching that I really enjoyed.

- I liked how hands on it was with group learning.
Miscellaneous....

1. Example of Preclass Readings

2. Where can I find rich, thought-provoking tasks?

(I still have a long way to go here in Calculus II.)
References


   Examples from Liberal Arts Math, Math for Elementary School Teachers, Calculus I


4. Joint work with Xianwei Y Van Harpen (UW-Milwaukee)

   Beginning teachers’ impressions on active learning (specifically the *Five Practices*)
Comments?

keith.nabb@uwrf.edu
Using the Five Practices in Mathematics Teaching (to appear in *Mathematics Teacher*)

Keith Nabb
Erick B. Hofacker
Kathryn T. Ernie
Susan Ahrendt

References (Possible Sources for Mathematically Rich Tasks)

**Elementary/Middle Grades**


**High School/College**


lessons.” *Mathematics Teacher* 105: 180-188.


Nabb, Keith, and Daniel Nghiem. “Reflections on good calculus questions from students and colleagues.” Presentation at the Annual Conference for the American Mathematical Association for Two-Year Colleges, New Orleans, LA, November 19-22, 2015.


**Websites (all grade levels)**

Engage NY: https://www.engageny.org/common-core-curriculum

Google: Dan Meyer’s Three-Act Math Tasks

http://map.mathshell.org/

Illustrativemathematics.org

Mathalicious.com

https://nrich.maths.org/

Openmiddle.com

Youcubed.org

Yummymath.com