Beyond Newton and Leibniz: The Making of Modern Calculus

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Calculus Before Newton & Leibniz

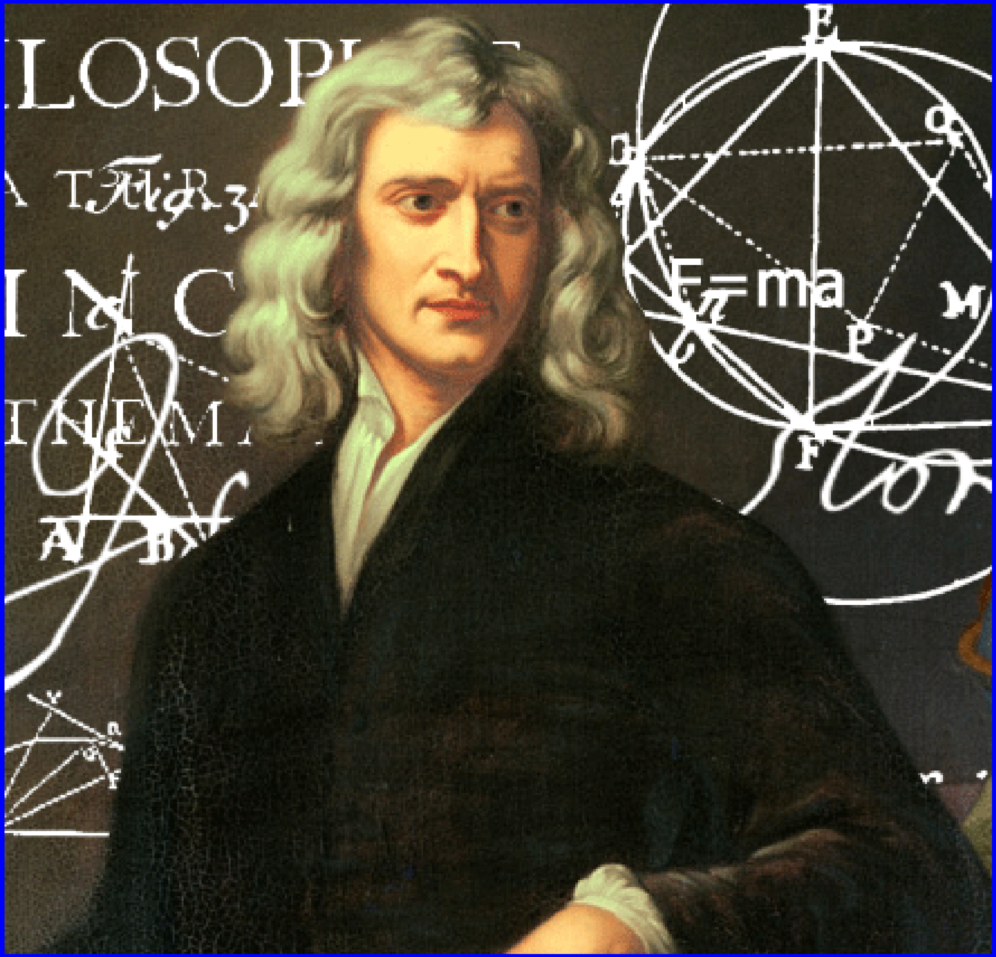
- Four Major Scientific Problems of the 17th Century:

1. Given distance as a function of time, find the velocity and acceleration, and conversely.
2. Find tangent line to a curve.
3. Find max and min value of a function.
4. Find the length of a curve, e.g. distance covered by a planet in a given interval of time

Early 17th Century Advances in Calculus

- Tangent lines to a curve: Roberval, Fermat, Descartes, Barrow.
- Areas, volumes, centers of gravity, lengths of curves began with Kepler.
- Galileo: motion problems, infinity
- Bonaventura Cavalieri (1598-1647): Method of indivisibles.
- John Wallis (1616-1703)

Isaac Newton



Gottfried Leibniz

Newton & Leibniz: Discovery and Controversy

- Newton (1642-1727): Method of fluxions
- 1687-publication of Principia Mathematica
    * fluent: a variable quantity
    * fluxion: its rate of change
- Leibniz (1646-1716)
    * 1684- published his papers on Calculus- based on discoveries he made dating back to 1673.

Comparison of Newton & Leibniz

- They both built Calculus on algebraic concepts.
- Their algebraic notation and techniques were more effective than the traditional geometric approach to concepts in Calculus.
- They reduced area and volume problems to antidifferentiation which were previously treated as summations.

- Newton used the infinitely small increments in x and y as a means of determining the fluxions (derivatives)---the limits of the ratios of the increments as they became smaller and smaller.
- Leibniz deals directly with the infinitely small increments in x and y, the differentials, and the relationship between them.
- Newton used series freely to represent functions; Leibniz used the closed form.

Background Leading to the Rigorization of Calculus

- 19th Century: “The Age of Rigor”---Analysis is given a foundation still considered today to be satisfactory.
- Rigorizing Calculus was substantially more than revising a few basic concepts and proofs of theorems.
- Produced new areas of mathematics, e.g. Point-set topology---which form the underpinnings of modern analysis.

Background

- Rigor was not the most important issue for 19th century mathematicians----majority worked on extending and applying previously developed theories
- the development of new technical theorems stimulated a growing interest in foundations, e.g., Fourier series challenged the old views of function, integral, continuity, convergence.

Background

- Teaching analysis motivated rigorizing analysis ---difficulty in attempting to explain vague concepts.
- the emancipation of mathematics from science increased the need for rigorizing Calculus.
- 18^{th} century analysis was closely connected to theoretical physics---Rules of analysis were validated by their success in physical situations, e.g. solutions of certain differential equations, sums

- 17th century: Analysis was based on geometry --an appendage to classical Greek geometry.
- 18th century: Attempts made to base analysis on algebra---rejected in 19th century.
- 19th century: the natural numbers and arithmetic became the building stones of Calculus: "the arithmetization of analysis"

Two Periods of the Rigorization of Analysis

- French Period: first half of the 19th century

  Leader: Augustin Cauchy

- German Period: second half of 19th century

  Leader: Karl Weierstrass

Functions

- Calculus became a study of functions from the time of Leonhard Euler (1707-1783)

- Euler’s Definitions of a Function

(a) An analytic expression (formula) containing constants and variables.

(b) A variable depending on another variable.

Functions

- Joseph Louis Lagrange (1736-1813) held the same views of functions as Euler

- Augustin-Louis Cauchy (1789-1857)---viewed functions exclusively as variables depending on other variables. (Cours d’Analyse—1821)

Functions

- Joseph Fourier (1768-1830)

  " The function f(x) represents a succession of values of ordinates each of which is arbitrary. An infinity of values being given to the abscissas x, there are an equal number of ordinates f(x). . . . We do not suppose these ordinates to be subject to a common law; they succeed each other in any manner whatever, and each of them is given as it were a single quantity."

Functions

- Peter Lejeune-Dirichlet (1805-1859)
  * his definition of a function is similar to that of Fourier and is essentially how functions are defined today.

- Functions were not always used the way they were defined.

Functions

- 18^{th} century mathematicians believed that a function must have the same analytic expression (formula) throughout.

- Late 18^{th} century: Euler & Lagrange allowed functions to have different expressions in different domains.

Functions

- Euler & Lagrange used the word “continuous” where the same expression held and the word “discontinuous” at points where the expression changed formula. (In modern sense, the entire function could be continuous.

Functions

- By the beginning of the 19th century, it was recognized that not all properties of algebraic functions could be extended to all functions which raised questions about what was really meant by a function, continuity, differentiability,and integrability.

Functions

- Cauchy categorized functions as either explicit or implicit---they are always represented by some equation or expression.
- Cauchy also divided functions into simple and composite functions.
- Many of Cauchy’s proofs and concepts did not rely on analytic expressions.
- Textbooks in early 19th century had varying definitions of functions and often deduced theorems which did not follow from the definition of a function.

Pioneers in the Reform of Calculus

Bernhard Bolzano (1781-1848)

- priest, philosopher, mathematician from Bohemia.
- his revolutionary work went unnoticed for approximately 50 years.
- his initial motivation was prompted by his attempt to give a purely arithmetical proof of the Fundamental Theorem of Algebra. (first proved by Gauss using a geometric approach)

Bernard Bolzano

Bolzano

- He denied the existence of infinitely small numbers (infinitesimals) and infinitely large numbers.
- 1817: gave a proper definition continuity:

  A function f(x) is continuous in an interval if at any x in the interval the difference $f(x + w) - f(x)$ can be made as small as one wishes by taking w sufficiently small.

Bolzano

- He was the first (1817) to define the derivative of f(x) as the quantity f ' (x) which the ratio

$$\frac{f(x+\Delta x)-f(x)}{\Delta x}$$

approaches indefinitely closely as Δx approaches 0 through positive and negative values.

Bolzano

- Most mathematicians in the early 19th century believed and “proved” that a continuous function must be differentiable except at isolated points.
- Bolzano understood the distinction between continuity and differentiability----in 1834 gave an example of a continuous function which has no finite derivative at any point. It was never published.

Bolzano

- Since this function (curve) did not have an analytic expression, it would not have been acceptable to the mathematicians of that era.
- Proved the Intermediate Value Theorem.
- 1851, his work on paradoxes of the infinite was published posthumously----he anticipated many of Georg Cantor’s ideas on infinite sets.

Augustin-Louis Cauchy

Augustin-Louis Cauchy

- His three major works were the pace-setter for the introduction of rigor into the field of analysis.

  * Cours d'Analyse (1821)

  * Resume des Lecons sur le calcul infinitesimal (1822)

  * Lecons sur le calcul differentiel (1829)

Cauchy

- His definitions of basic Calculus concepts appear to be very wordy, vague, and non-rigorous----no epsilons, deltas, quantifiers, inequalities.
- the rigor, including notation, surfaces in his proofs.
- His work was the first comprehensive treatment of analysis to be based on a clear definition of the limit concept.

Cauchy's Definition of a Limit

"When the successive values attributed to a variable approach indefinitely a fixed value so as to end by differing from it by as little as one wishes, this last (fixed value) is called the limit of all the others. Thus, for example, an irrational number is the limit of diverse fractions which furnish more and more approximate values of it."

How Cauchy Reconciled Rigor With Infinitesimals

- Traditionally, infinitesimals were infinitely small fixed numbers.
- Cauchy defined an infinitesimal to be a variable with zero as its limit.
- Cauchy's words:

"One says that a variable quantity becomes infinitely small when its numerical value decreases indefinitely in such a way as to converge towards the value zero."

Cauchy

- Prior to Cauchy, previous expositions of the Calculus generally adopted the differential as the fundamental concept.
- The derivative was introduced as the "differential coefficient in the expression

  $dy = f'(x)\, dx$

- Cauchy viewed the derivative as the limit of a difference quotient.

Cauchy

- Cauchy is credited with introducing the Chain Rule to find the derivative of the composition of two functions.
- Three major features of Cauchy’s work that set the pattern for future expositions of the Calculus:

1. His explicit formulation of continuity and differentiability in terms of limits.

Cauchy

2. The central role that he accorded to the Mean Value Theorem (known previously to Lagrange, but not used much by him)
3. His definition of the definite integral and proof of the Fundamental Theorem of Calculus.

- During the 18th century, the integral was generally regarded simply as the inverse of the derivative.

Karl Weierstrass

Karl Weierstrass (1815-1897)

- The Calculus of Newton & Leibniz was a calculus of geometric variables---quantities associated with a geometric curve.
- Euler, Lagrange, & Cauchy attempted to substitute the principles of arithmetic in the foundations of Calculus----only partially successful.
- Real numbers were only understood intuitively---irrational numbers can be approximated by rational numbers.

Weierstrass

- Richard Dedekind (1831-1916) claimed that $\sqrt{2}\cdot\sqrt{3}$ had never been rigorously proved.
- Weierstrass was the first to give a rigorous construction of the real number system.
- Weierstrass is credited with the modern-day epsilon-delta definition of a limit.

Weierstrass

- $$\lim_{x \to a} f(x) = L \; if \; for \; \forall \; \varepsilon > 0 \; \exists \delta > 0$$
  $$such \; that \; |f(x) - L| < \varepsilon \; whenever$$
  $$0 < |x - a| < \delta.$$

Weierstrass

- Bolzano-Weierstrass Theorem: Every bounded montonic sequence is convergent.
- Weierstrass found a function which is everywhere continuous, but nowhere differentiable.

Weierstrass

- $$f(x) = \sum_{n=0}^{\infty} b^n \cos(a^n \pi x)$$